Published by Institute of Physics Publishing for SISSA/ISAS
Received: *May 6, 2003*
Revised: *May 23, 2003*
Accepted: *May 28, 2003*

Current-current deformations of conformal field theories, and WZW models

Stefan Förste[a] and Daniel Roggenkamp[ab]

[a] *Physikalisches Institut, Universität Bonn*
Nussallee 12, D-53115 Bonn, Germany
[b] *Department of Mathematics, King's College London*
Strand, London WC2R 2LS, United Kingdom
E-mail: forste@th.physik.uni-bonn.de, roggenka@th.physik.uni-bonn.de

Abstract: Moduli spaces of conformal field theories corresponding to current-current deformations are discussed. For WZW models, CFT and sigma model considerations are compared. It is shown that current-current deformed WZW models have WZW-like sigma model descriptions with non-bi-invariant metrics, additional B-fields and a non-trivial dilaton.

Keywords: Sigma Models, Conformal Field Models in String Theory, Conformal and W Symmetry.

JHEP05(2003)071

http://jhep.sissa.it/archive/papers/jhep052003071/jhep052003071.pdf

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1. Introduction

Moduli spaces of conformal field theories are important objects in the study of two dimensional quantum field theories because they describe critical subspaces in the space of coupling constants. In string theories, whose small coupling limits are described by conformal field theories, these moduli spaces arise as parameter spaces of string vacua. The understanding of moduli spaces of conformal field theories is thus a very important issue for string theory.

Although conformal field theories are quite well understood, there are only few examples of explicitly known moduli spaces at present, most of which correspond to free field theories as e.g. toroidal conformal field theories or orbifolds thereof.

The reason for this is that a good conceptual understanding of deformations of conformal field theories beyond conformal perturbation theory is still lacking. Perturbation theory is usually technically involved, at least when one wants to obtain higher order contributions, and hence is only applicable to study CFT moduli spaces in small neighbourhoods of explicitly known models. In particular it is in general not possible to obtain global information about the moduli spaces from perturbation theory, except in situations, where symmetry as e.g. supersymmetry is preserved by the corresponding deformations.

In this article, special deformations of conformal field theories, namely those generated by perturbations with products of holomorphic and antiholomorphic currents are studied. As was shown in [1], such deformations preserve the algebras of the perturbing fields. Indeed, this gives enough structure to determine global properties of the subspaces of moduli spaces corresponding to this kind of deformations.

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Important examples of non-free conformal field theories admitting current-current deformations are WZW models (see e.g. [2]). The corresponding moduli spaces for WZW models associated to compact semi-simple Lie groups, will be discussed in detail.

Since WZW models have descriptions as sigma models on Lie groups, it is a natural question if there are such descriptions for all conformal field theories from these moduli spaces. In fact, families of sigma models containing WZW models at special points have been discussed by many authors (e.g. in [3]–[7]). In particular one-parameter families of sigma models containing the SU(2)-WZW models have been studied very explicitly by Giveon and Kiritsis [4], who also compared them to the families of current-current deformed SU(2)-WZW models which were described in [8]. Ideas about a generalization of these considerations to arbitrary WZW models have also been presented in [4, 6, 9].

In this article, WZW-like sigma model representations of current-current deformed WZW models will be explicitly constructed. These are sigma models with the same target space as the "undeformed" WZW model in the family, but with different (in general not bi-invariant) metrics and additional B-fields.

Thus, we obtain explicit descriptions of the moduli spaces of current-current deformed WZW models associated to compact semi-simple Lie groups in terms of conformal field theories as well as in terms of sigma models.

In section 2, exactly marginal current-current deformations of conformal field theories are discussed. We start from the facts obtained in [1], that perturbations of conformal field theories with products of holomorphic and antiholomorphic currents are exactly marginal iff the holomorphic as well as the antiholomorphic currents belong to commutative current algebras, and that in this case these holomorphic and antiholomorphic current algebras are preserved under the deformations. This can be used to reduce the problem of studying finite current-current deformations to first order deformation theory, which is carried out in appendix A, and from which it follows that the effect of these deformations on the CFT structures is completely captured by pseudo orthogonal transformations of their charge lattices with respect to the preserved commutative current algebras. The corresponding deformation spaces can thus be described by

$$\mathcal{D} \simeq \mathrm{O}(d,\bar{d})/\mathrm{O}(d) \times \mathrm{O}(\bar{d})\,. \tag{1.1}$$

This generalizes the deformation results for toroidal conformal field theories [10]. The corresponding moduli spaces are obtained from the deformation spaces by taking the quotients with respect to "duality groups".

In section 3 we discuss an important class of examples, namely WZW models corresponding to compact semi-simple Lie-groups. The general results from section 2 are compared to a realization of deformed WZW models obtained from a representation of WZW models as orbifolds of products of generalized parafermionic and toroidal models given in [11].

In section 4 we discuss various aspects of exactly marginal deformations of WZW models from a sigma model perspective. This approach is best suited for a semiclassical treatment and in that sense less powerful than the algebraic one. However, it can illustrate the results and provide a picture for the class of deformed models. Exactly marginal

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deformations of WZW models from the sigma model perspective have been discussed in the past mainly for rank one groups [3]–[5] and for models where coordinates can be chosen such that the relevant set of chiral and anti-chiral currents follows manifestly from the equations of motion [12].

Mimicking the orbifold realization of deformed WZW models described in section 3, we will consider an orbifold of a direct product consisting of a vectorially gauged WZW model and a d-dimensional torus model, where d is the rank of the group. Since a sigma model is not very well designed to accommodate orbifolds, we perform an axial-vector duality (generalized T-duality) to obtain a dual description without an additional orbifold action. To this end, we first implement the orbifold by gauging in addition an axial symmetry of the WZW model combined with shifts in the torus factor. We force the corresponding gauge bundle to be flat but choose the zero modes of the corresponding Lagrange multiplier such that the gauge bundle is twisted in a non-trivial way. It turns out that integrating out the gauge bundle instead of the Lagrange multiplier provides a sigma model without an additional orbifold action. The result is a “WZW-like” model, i.e. a sigma model with Lie group as target space, and an action given by a WZW action in which the bi-invariant metric is replaced by a more general bilinear form which is neither bi-invariant nor necessarily symmetric.

The same sigma models can be obtained as coset models of products of the original WZW models and d-dimensional torus models with gauge group $\mathrm{U}(1)^d$, embedded into both factors.

All the sigma model manipulations described so far are carried out at a classical level. Quantum mechanically one should replace the procedure of solving equations of motion by performing gaussian integrals. These in general provide functional determinants, which in turn generate a non-trivial dilaton. In a pragmatic approach this can be computed by imposing conformal invariance, i.e. requiring vanishing beta functions. We will use a more elegant way consisting of a comparison of the hamiltonian of the model and a generalized laplacian, which depends on the dilaton [13].

Finally, we illustrate some of the results in the example of the deformed SU(2)-WZW model.

We should mention that many of the employed techniques can be found in the literature. The corresponding references will be given in the text.

2. Current-current deformations of CFTs

Although not proven in general, it is widely believed that all deformations of conformal field theories are generated by perturbations of the theories with marginal fields, i.e. fields $\mathcal{O}_i$ with conformal weights $h(\mathcal{O}_i) = 1 = \overline{h}(\mathcal{O}_i)$.

The perturbed correlation functions on a conformal surface Σ of a combination of operators $X(p_1, \dots, p_k)$, $p_i \in \Sigma$ are defined to be

$$\langle X(p_1, \dots, p_k) \rangle_\Sigma^\lambda := \left\langle X(p_1, \dots, p_k) \exp\left(\sum_i \lambda_i \int_\Sigma \mathcal{O}_i d\mu_\Sigma \right) \right\rangle_\Sigma , \tag{2.1}$$

where the integrals have to be regularized due to the appearance of singularities. If the perturbed correlation functions define a quantum field theory, which is a fixed point of the renormalization group flow, it is again a conformal field theory. This however is not the case in general. Indeed, preservation of conformal invariance by perturbations gives non linear restrictions on the fields, which generate it (see e.g. [10]). This means that the set of exactly marginal fields, i.e. those which generate deformations of conformal field theories are not vector spaces in general. In particular the deformation spaces of conformal field theories need not necessarily be manifolds but may have singularities. (For more details on conformal deformation theory see e.g. [10] and [14]–[16]).

In the following, deformations of conformal field theories generated by a special class of marginal fields, namely products of holomorphic and antiholomorphic currents will be discussed. These are simple enough to give a global description of the deformation spaces corresponding to them and to express the data of deformed CFTs explicitly in terms of the data of the undeformed ones.

We consider conformal field theories, whose holomorphic and antiholomorphic W-algebras contain current algebras $\hat{\mathfrak{g}}_k$, $\hat{\overline{\mathfrak{g}}}_{\overline{k}}$ corresponding to Lie algebras $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ and $k, \overline{k} \in \mathbb{N}$, i.e. for every $j, j' \in \mathfrak{g}$ there exist holomorphic fields $j(z), j'(z)$ of conformal weight $h = 1$ in the theory, such that

$$\begin{aligned} j(z)j'(w) &= \frac{kK_{\mathfrak{g}}(j,j')}{(z-w)^2} + \frac{[j,j'](w)}{(z-w)} + \mathrm{reg} \\ T(z)j(w) &= \frac{j(w)}{(z-w)^2} + \frac{\partial j(w)}{(z-w)} + \mathrm{reg}\,, \end{aligned} \tag{2.2}$$

where $K_{\mathfrak{g}}(.,.)$ is a bi-invariant scalar product on $\mathfrak{g}$ and $[.,.]$ its Lie bracket. The holomorphic energy momentum tensor $T(z)$ can be written as

$$T(z) = T_0(z) + \frac{1}{2(k+\check{g})} \sum_{\alpha,\beta} \kappa^g_{\alpha\beta} :j^\alpha(z) j^\beta(z):\,, \qquad \kappa^{\mathfrak{g}} := K_{\mathfrak{g}}^{-1}\,, \tag{2.3}$$

with $\check{g}$ the dual Coxeter number of $\mathfrak{g}$, $(j^\alpha)_\alpha$ a basis of $\mathfrak{g}$ and $T_0(z)j(w) = \mathrm{reg}$. The same holds for the antiholomorphic current algebra with $(\mathfrak{g}, k)$ replaced by $(\bar{\mathfrak{g}}, \bar{k})$.

In particular, there is a subspace of the CFT Hilbert space isomorphic to $\mathfrak{g} \otimes \overline{\mathfrak{g}}$ of marginal fields. However, not all of these fields are exactly marginal. As was shown in [1], such fields are exactly marginal if and only if, under the isomorphism above, they correspond to elements of $\mathfrak{a} \otimes \overline{\mathfrak{a}}$, for any abelian subalgebras $\mathfrak{a} \subset \mathfrak{g}$, $\overline{\mathfrak{a}} \subset \overline{\mathfrak{g}}$. Moreover deformations generated by these exactly marginal fields preserve the current algebras $\hat{\mathfrak{a}}$, $\hat{\overline{\mathfrak{a}}}$ corresponding to $\mathfrak{a}$, $\overline{\mathfrak{a}}$.

Hence, every pair of abelian $\hat{u}(1)^d \simeq \mathfrak{a} \subset \mathfrak{g}$, $\hat{\overline{u}}(1)^{\overline{d}} \simeq \overline{\mathfrak{a}} \subset \overline{\mathfrak{g}}$ gives rise to a family of conformal field theories with current algebras $\hat{\mathfrak{a}}$, $\hat{\overline{\mathfrak{a}}}$. All these families meet in the original model. Since conformal field theories are invariant under automorphisms of their holomorphic and antiholomorphic W-algebras, pairs of abelian subalgebras which are identified under such automorphisms give rise to equivalent deformations.

We assume in the following that the conformal field theory is unitary, and its Hilbert space decomposes into tensor products of irreducible highest weight representations $\mathcal{V}_Q$, $\overline{\mathcal{V}}_{\overline{Q}}$

of $\hat{\mathfrak{a}}$, $\hat{\overline{\mathfrak{a}}}$, which are characterized by their charges $Q \in \mathfrak{a}^*$, $\overline{Q} \in \overline{\mathfrak{a}}^*$, and whose lowest conformal weights are given by $h_Q = \frac{1}{2}\kappa(Q,Q)$, $\overline{h} = \frac{1}{2}\overline{\kappa}(\overline{Q},\overline{Q})$, as can be read off from (2.2):[1]

$$\mathcal{H} \simeq \bigoplus_{(Q,\overline{Q})\in\Lambda} \mathcal{H}_{Q\overline{Q}} \otimes \mathcal{V}_Q \otimes \overline{\mathcal{V}}_{\overline{Q}}\,. \tag{2.4}$$

The set of charges $\Lambda \subset \mathfrak{a}^* \times \overline{\mathfrak{a}}^*$ forms a lattice equipped with bilinear pairing $\langle .,.\rangle := \kappa - \overline{\kappa}$.

As is shown in appendix A, deformations corresponding to pairs $(\mathfrak{a},\overline{\mathfrak{a}})$ only affect the representation theory of the W-algebras $\hat{\mathfrak{a}}$, $\hat{\overline{\mathfrak{a}}}$, but not the OPE-coefficients of $\hat{\mathfrak{a}} \times \hat{\overline{\mathfrak{a}}}$-highest weight vectors. More precisely, if one chooses suitable connections on the bundles of Hilbert spaces over the deformation spaces [16], the effect of the deformations on the CFT structures is completely captured by transformations of the charge lattices Λ in the identity component $\mathrm{O}(d,\overline{d})_0$ of the pseudo orthogonal group $\mathrm{O}(d,\overline{d})$, and all structures independent of the charges are parallel with respect to the chosen connection.

That these $\mathrm{O}(d,\overline{d})_0$-transformations indeed give rise to new modular invariant partition functions and also preserve locality is easy to see even without any perturbation theory. Locality is maintained because of the preservation of $\kappa - \overline{\kappa}$ by transformations $O \in \mathrm{O}(d,\overline{d})$. To show the preservation of modular invariance we consider the torus partition function depending on modular parameters τ and $\overline{\tau}$ ($q = e^{2\pi i\tau}$, $\overline{q} = e^{-2\pi i\overline{\tau}}$) of the $O(t)$-transformed model along a smooth path $O : [-1,1] \to \mathrm{O}(d,\overline{d})_0$ with $O(0) = 1$, $\partial_t O(t)|_{t=0} = T \in o(d,\overline{d})$

$$\begin{aligned}
Z^{O(t)}(q,\overline{q}) &:= \mathrm{tr}_{\mathcal{H}}\left(q^{L_0^{O(t)}-\frac{c}{12}}\overline{q}^{\overline{L}_0^{O(t)}-\frac{c}{12}}\right)\\
&= \mathrm{tr}_{\mathcal{H}}\Big(q^{L_0-\frac{c}{12}}\overline{q}^{\overline{L}_0-\frac{c}{12}}e^{\pi i\,\mathrm{Re}\,(\tau)(O(t)^*-1)\langle .,.\rangle((j_0,\overline{\jmath}_0),(j_0,\overline{\jmath}_0))}\times\\
&\qquad\qquad \times e^{-\pi\,\mathrm{Im}\,(\tau)(O(t)^*-1)\kappa\oplus\overline{\kappa}((j_0,\overline{\jmath}_0),(j_0,\overline{\jmath}_0))}\Big)\\
&= \mathrm{tr}_{\mathcal{H}}\Big(q^{L_0-\frac{c}{12}}\overline{q}^{\overline{L}_0-\frac{c}{12}}e^{-\pi\,\mathrm{Im}\,(\tau)(O(t)^*-1)\kappa\oplus\overline{\kappa}((j_0,\overline{\jmath}_0),(j_0,\overline{\jmath}_0))}\Big)
\end{aligned} \tag{2.5}$$

and use the modular transformation properties of the unspecialized characters:

$$\begin{aligned}
\partial_t Z^{O(t)}(q,\overline{q})|_{t=0} &= \mathrm{Im}\,(\tau)T_{\alpha\overline{\alpha}}\mathrm{tr}_{\mathcal{H}}\left(j_0^{\alpha}\overline{\jmath}_0^{\overline{\alpha}}q^{L_0-\frac{c}{12}}\overline{q}^{\overline{L}_0-\frac{c}{12}}\right)\\
&= \frac{-\mathrm{Im}\,(\tau)}{4\pi^2}T_{\alpha\overline{\alpha}}\partial_{z_\alpha}\partial_{\overline{z}_{\overline{\alpha}}}|_{z=\overline{z}=0}\mathrm{tr}_{\mathcal{H}}\left(q^{L_0-\frac{c}{12}}\overline{q}^{\overline{L}_0-\frac{c}{12}}e^{2\pi i\kappa(z,j_0)}e^{-2\pi i\overline{\kappa}(\overline{z},\overline{\jmath}_0)}\right)\\
&= \frac{-\mathrm{Im}\,(\tau)}{4\pi^2}T_{\alpha\overline{\alpha}}\partial_{z_\alpha}\partial_{\overline{z}_{\overline{\alpha}}}|_{z=\overline{z}=0}\mathrm{tr}_{\mathcal{H}}\left(\tilde{q}^{L_0-\frac{c}{12}}\tilde{\overline{q}}^{\overline{L}_0-\frac{c}{12}}e^{2\pi i\kappa(\frac{z}{\tau},j_0)}e^{-2\pi i\overline{\kappa}(\frac{\overline{z}}{\overline{\tau}},\overline{\jmath}_0)}\right)\\
&= \mathrm{Im}\,(-\frac{1}{\tau})T_{\alpha\overline{\alpha}}\mathrm{tr}_{\mathcal{H}}\left(j_0^{\alpha}\overline{\jmath}_0^{\overline{\alpha}}\tilde{q}^{L_0-\frac{c}{12}}\tilde{\overline{q}}^{\overline{L}_0-\frac{c}{12}}\right)\\
&= \partial_t Z^{O(t)}(\tilde{q},\tilde{\overline{q}})|_{t=0}\,,
\end{aligned} \tag{2.6}$$

where $\tilde{q} = e^{\frac{-2\pi i}{\tau}}$, $\tilde{\overline{q}} = e^{\frac{2\pi i}{\overline{\tau}}}$. This shows that invariance under the modular transformation $\tau \mapsto -\frac{1}{\tau}$ is preserved under $\mathrm{O}(d,\overline{d})_0$-transformations of the charge lattices. A similar calculation proves the statement for $\tau \mapsto \tau + 1$.

Since charges $(Q,\overline{Q})$ only characterize the $\hat{\mathfrak{a}} \times \hat{\overline{\mathfrak{a}}}$-modules up to automorphisms of the underlying Lie algebras, transformations of Λ by $\mathrm{O}(d) \times \mathrm{O}(\overline{d}) \subset \mathrm{O}(d,\overline{d})$ leave the

[1] From now on $\kappa = \frac{1}{k}\kappa^{\mathfrak{g}}|_{\mathfrak{a}}$, $\overline{\kappa} = \frac{1}{k}\kappa^{\overline{\mathfrak{g}}}|_{\overline{\mathfrak{a}}}$.

conformal field theory invariant. Thus, the deformation spaces corresponding to pairs $(a,\overline{a})$ are given by

$$\mathcal{D}_{(a,\overline{a})} \simeq \mathrm{O}(d,\overline{d})_0/((\mathrm{O}(d)\times\mathrm{O}(\overline{d}))\cap\mathrm{O}(d,\overline{d})_0) \simeq \mathrm{O}(d,\overline{d})/\mathrm{O}(d)\times\mathrm{O}(\overline{d})\,. \tag{2.7}$$

To get the respective moduli spaces from these deformation spaces, one has to identify points describing equivalent conformal field theories. In fact, the conformal field theories are specified by the charge lattices marked with the Hilbert spaces $\mathcal{H}_{(Q,\overline{Q})}$ of $\hat{\mathfrak{a}}\times\hat{\overline{\mathfrak{a}}}$-highest weight states (with all the structure they carry, as e.g. structure of modules of the Virasoro algebra etc.) and the coefficients of the operator product expansion of these highest weight states. Denoting these additional structures by S, and the automorphisms of Λ together with S by $\mathrm{Aut}(\Lambda,S)$, the components of the moduli spaces corresponding to $(\mathfrak{a},\overline{\mathfrak{a}})$ deformations can be written as

$$\mathcal{M}_{(\mathfrak{a},\overline{\mathfrak{a}})} \simeq \mathrm{Aut}(\Lambda,S)\backslash\mathrm{O}(d,\overline{d})/\mathrm{O}(d)\times\mathrm{O}(\overline{d})\,. \tag{2.8}$$

If the action of $\mathrm{Aut}(\Lambda,S)$ has fixed points, $\mathcal{M}_{(\mathfrak{a},\overline{\mathfrak{a}})}$ has singularities and the Hilbert space bundle over it has non-trivial monodromies around them. More precisely elements in $\mathrm{Aut}(\Lambda,S)$ act on the Hilbert space bundles over the deformation spaces, and monodromies around fixed points are given by the respective actions of the stabilizers.

This gives a very explicit description of the components of moduli spaces of conformal field theories corresponding to current-current deformations. In particular conformal field theories as above come in $\mathcal{D}_{(\mathfrak{a},\overline{\mathfrak{a}})}$ families of explicitly known CFTs, and the conformal field theory data at any point in these families can be easily reconstructed from the CFT data at one point.

A well known example of this kind is the moduli space of toroidal conformal field theories. These models have holomorphic and antiholomorphic W-algebras, each of which contains a $\hat{u}(1)^d$ current-algebra. They are completely characterized by their charge lattices ($\mathcal{H}_{Q,\overline{Q}}$ are one-dimensional for all charges), which for integer spin of the fields, locality and modular invariance of the torus partition function have to be even, integral, selfdual lattices of signature (d,d) in $\mathbb{R}^{d,d}$ [17, 18]. Hence, S is trivial and $\mathrm{Aut}(\Lambda,S)\simeq\mathrm{Aut}(\Lambda)\simeq\mathrm{O}(d,d,\mathbb{Z})$, such that the moduli spaces $\mathcal{M}_{d,d}$ corresponding to the current-current deformations are isomorphic to the Narain moduli spaces [18]

$$\mathcal{M}_d^{\mathrm{Narain}} \simeq \mathrm{O}(d,d,\mathbb{Z})\backslash\mathrm{O}(d,d)/\mathrm{O}(d)\times\mathrm{O}(d)\,, \tag{2.9}$$

of even, integral selfdual lattices of signature (d,d) in $\mathbb{R}^{d,d}$. This is the entire moduli space of toroidal conformal field theories.

A more general class of conformal field theories with current algebras are WZW models. These are discussed in the next section.

3. Deformations of WZW models

WZW models are conformal field theories associated to Lie groups G with bi-invariant metrics $\langle .,.\rangle$ (see e.g. [2]). For simplicity we will only consider compact semi-simple G with bi-invariant metrics corresponding to the Killing forms on the respective Lie algebras here.

So, let $k \in \mathbb{N}$, G a semi-simple Lie group, $\mathfrak{g}$ its Lie algebra of rank d with Killing form K/k, roots Δ, weights Ω, root lattice $\mathrm{Q}(\mathfrak{g})$, long root lattice $\mathrm{Q}_l(\mathfrak{g}) \subset \mathrm{Q}(\mathfrak{g})$, and weight lattice $\mathrm{P}(\mathfrak{g})$. Furthermore denote by $\hat{\Omega}_k$ the set of integrable weights of the affine Lie algebra $\hat{\mathfrak{g}}_k$ at level k.

The WZW models associated to (G, K, k) have the affine Lie algebras $\hat{\mathfrak{g}}_k$ as holomorphic and antiholomorphic W-algebras. Its Hilbert spaces decompose into tensor products of integrable highest weight representations $\mathcal{V}_{\hat{\lambda}}$, $\hat{\lambda} \in \hat{\Omega}_k$ of $\hat{\mathfrak{g}}_k$. For simplicity only diagonal WZW models are considered in the following, i.e. those WZW models whose Hilbert spaces are given by

$$\mathcal{H} \simeq \bigoplus_{\hat{\lambda} \in \hat{\Omega}_k} \mathcal{V}_{\hat{\lambda}} \otimes \overline{\mathcal{V}}_{\hat{\lambda}} \,. \tag{3.1}$$

Generically, the only marginal fields in WZW models are products of holomorphic and antiholomorphic currents from the current algebras. From the general considerations in section 2 it is clear that every pair of Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$, $\overline{\mathfrak{h}} \subset \mathfrak{g}$ gives rise to deformations of the WZW models. However, all such pairs lead to equivalent deformations, because all maximal abelian subalgebras of a semi-simple Lie algebra are pairwise conjugated.[2] Thus the deformation space of current-current-deformed WZW models is given by

$$\mathcal{D}_{\mathrm{WZW}} \simeq \mathrm{O}(d,d)/\mathrm{O}(d) \times \mathrm{O}(d) \,. \tag{3.2}$$

For a given Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ the integrable $\hat{\mathfrak{g}}_k$-highest weight representations decompose into $\hat{\mathfrak{h}}$-highest weight modules $\mathcal{V}_Q$, $Q \in \mathfrak{h}^*$ as follows

$$\mathcal{V}_{\hat{\lambda}} \simeq \bigoplus_{\mu \in \Gamma_k} \mathcal{V}_{\hat{\lambda},\mu} \otimes \mathcal{V}^{\mathrm{ext}}_{\mu} \simeq \bigoplus_{\mu \in \Gamma_k} \mathcal{V}_{\hat{\lambda},\mu} \otimes \bigoplus_{\delta \in \mathrm{Q}_l(\mathfrak{g})} \mathcal{V}_{(\mu + k\delta)} \,, \tag{3.3}$$

where $\Gamma_k := \mathrm{P}(\mathfrak{g})/k\mathrm{Q}_l(\mathfrak{g})$ is a finite abelian group, $\mathcal{V}_{\hat{\lambda},\mu}$ is a highest weight module of a generalized parafermionic W-algebra associated to the coset construction $\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}$, and $\mathcal{V}^{\mathrm{ext}}_{\mu}$ are highest weight modules with respect to an extended $\hat{\mathfrak{h}}$ W-algebra [11].[3]

From this, the charge lattice can be read off to be

$$\Lambda_0 = \{(\mu, \overline{\mu}) \in \mathrm{P}(\mathfrak{g}) \times \mathrm{P}(\mathfrak{g}) \,|\, \mu - \overline{\mu} \in \mathrm{Q}(\mathfrak{g})\} \,. \tag{3.4}$$

The "duality group" is given by the automorphisms of Λ_0 compatible with the additional structures S_k, alluded to in the last section. In the case of diagonal WZW models discussed here, all these structure are determined by representation theory, and the duality group is given by the semi-direct product

$$\mathrm{Aut}(\Lambda_0, S_k) \simeq \mathrm{A}(\mathfrak{g}) \ltimes \mathrm{W}(\mathfrak{g}) \tag{3.5}$$

[2]This is not true for non-semi-simple Lie algebras, where one gets more interesting moduli spaces.

[3]In terms of characters $\chi_{\hat{\lambda}}(q,w)$ corresponding to the $\hat{\mathfrak{g}}_k$-highest weight representations, this is just the string function decomposition [19]

$$\chi_{\hat{\lambda}}(q,w) = \sum_{\mu \in \Gamma_k} c^{\hat{\lambda}}_{\mu}(q) \theta_{\mu}(q,w) \,, \qquad \theta_{\mu}(q,w) := \sum_{\delta \in \mathrm{Q}_l(\mathfrak{g})} q^{\frac{1}{2}\kappa(\mu+k\delta, \mu+k\delta)} e^{2\pi i w(\mu+k\delta)} \,,$$

with $c^{\hat{\lambda}}_{\mu}$ the string functions and η the Dedekind-η-function.

of the automorphism group $\mathrm{A}(\mathfrak{g})$ of the root lattice $\mathrm{Q}(\mathfrak{g})$ with the Weyl group $\mathrm{W}(\mathfrak{g})$, where $\mathrm{A}(\mathfrak{g})$ acts diagonally on $\Lambda_0 \subset \mathrm{P}(\mathfrak{g}) \times \mathrm{P}(\mathfrak{g})$, and $\mathrm{W}(\mathfrak{g})$ acts on the second factor only (see [6] for a discussion of dualities of WZW models). Since $\mathrm{A}(\mathfrak{g}) \simeq \mathrm{W}(\mathfrak{g}) \ltimes \mathrm{F}(\mathfrak{g})$, the duality group can be written as

$$\mathrm{Aut}(\Lambda_0, S_k) \simeq (\mathrm{W}(\mathfrak{g}) \times \mathrm{W}(\mathfrak{g})) \ltimes \mathrm{F}(\mathfrak{g})\,, \tag{3.6}$$

with the Weyl groups acting separately on the two factors of $\Omega \times \Omega$ and $\mathrm{F}(\mathfrak{g})$ acting diagonally. Note that this group is finite, as opposed to the "duality groups" of d-dimensional toroidal models for $d > 1$. For the special case $\mathfrak{g} = \mathrm{su}(2)$, the group (3.6) and the toroidal duality group $\mathrm{O}(1,1,\mathbb{Z})$ coincide.

Given the duality group, the moduli space of current-current deformed WZW models can now be written as (2.9)

$$\mathcal{M}_{\mathrm{WZW}} \simeq (\mathrm{W}(\mathfrak{g}) \times \mathrm{W}(\mathfrak{g})) \ltimes \mathrm{F}(\mathfrak{g}) \backslash \mathrm{O}(d,d)/\mathrm{O}(d) \times \mathrm{O}(d)\,. \tag{3.7}$$

In fact, Λ_0 has an even integral selfdual sublattice

$$\Lambda_k := \{(\mu, \overline{\mu}) \in \mathrm{P}(\mathfrak{g}) \times \mathrm{P}(\mathfrak{g}) \,|\, \mu - \overline{\mu} \in k\mathrm{Q}_l(\mathfrak{g})\} \subset \Lambda_0 \tag{3.8}$$

of signature (d,d), which can be regarded as charge lattice of d-dimensional toroidal conformal field theory.

Let us denote by $\mathrm{Aut}(\Lambda_k, S_k) \subset \mathrm{Aut}(\Lambda_0, S_k)$ the subgroup of the duality group fixing Λ_k. This group is also a subgroup of $\mathrm{O}(d,d,\mathbb{Z})$.

Every even integral selfdual signature (d,d) sublattice of Λ_0, which is obtained from Λ_k by applying a transformation of $\mathrm{Aut}(\Lambda_0, S_k)$ gives rise to a representation of the WZW model as orbifold model

$$\hat{\mathfrak{g}}_k \simeq \left(\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}} \otimes \mathrm{t}_{\Lambda_\mathrm{k}}\right)/\Gamma_k\,. \tag{3.9}$$

of a product of a coset model $\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}$ and a toroidal conformal field theory t_{Λ_k} with charge lattices Λ_k. Such representations were presented in [11] and were actually used in [9] in the study of dualities of WZW and coset models.

The torus partition function of coset and toroidal models (with the notation from footnote 3) are given by

$$Z^{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}}(q,\overline{q}) = \sum_{\hat{\lambda} \in \hat{\Omega}_k} \sum_{\mu \in \Gamma_k} |\eta(q)|^{2d} c^{\hat{\lambda}}_{\mu}(q) \overline{c}^{\hat{\lambda}}_{\mu}(\overline{q})\,, \tag{3.10}$$

$$Z^{\mathrm{t}_{\Lambda_k}}(q,\overline{q}) = \sum_{\mu \in \Gamma_k} \frac{\theta_\mu(q)\overline{\theta}_\mu(\overline{q})}{|\eta(q)|^{2d}} = \sum_{(\mu,\overline{\mu}) \in \Lambda} \frac{q^{\frac{\mu^2}{2k}} \overline{q}^{\frac{\overline{\mu}^2}{2k}}}{|\eta(q)|^{2d}}\,. \tag{3.11}$$

Γ_k acts on the Hilbert spaces of the models as follows

$$\begin{aligned} \gamma \in \Gamma_k : \quad \gamma|_{\mathcal{V}_{\hat{\lambda},\mu} \otimes \overline{\mathcal{V}}_{\hat{\lambda},\overline{\mu}}} &= e^{\pi i \kappa(\gamma, \mu + \overline{\mu})} \mathrm{id}_{\mathcal{V}_{\hat{\lambda},\mu} \otimes \overline{\mathcal{V}}_{\hat{\lambda},\overline{\mu}}}\,, \\ \gamma|_{\mathcal{V}^{\mathrm{ext}}_{\mu} \otimes \overline{\mathcal{V}}^{\mathrm{ext}}_{\overline{\mu}}} &= e^{-\pi i \kappa(\gamma, \mu + \overline{\mu})} \mathrm{id}_{\mathcal{V}^{\mathrm{ext}}_{\mu} \otimes \overline{\mathcal{V}}^{\mathrm{ext}}_{\overline{\mu}}}\,, \end{aligned} \tag{3.12}$$

giving rise to the (α,β)-twisted torus partition functions for $\alpha,\beta\in\Gamma_k$

$$Z^{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}}_{\alpha,\beta}(q,\overline{q}) = \sum_{\hat\lambda\in\hat\Omega_k}\sum_{\mu\in\Gamma_k} e^{\pi i\kappa(\alpha,2\mu-\beta)}|\eta(q)|^{2d} c^{\hat\lambda}_{\mu}(q)\overline{c}^{\hat\lambda}_{\mu-\beta}(\overline{q})\,, \tag{3.13}$$

$$Z^{\mathrm{t}_\Lambda}_{\alpha,\beta}(q,\overline{q}) = \sum_{\mu\in\Gamma_k} e^{-\pi i\kappa(\alpha,2\mu-\beta)}\frac{\theta_\mu(q)\overline{\theta}_{\mu-\beta}(\overline{q})}{|\eta(q)|^{2d}}\,. \tag{3.14}$$

From this, one can easily read off that the orbifold partition function of (3.9)

$$Z^{\mathrm{orb}}(q,\overline{q}) = \frac{1}{|\Gamma_k|}\sum_{\alpha,\beta\in\Gamma_k} Z^{\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}}_{\alpha,\beta}(q,\overline{q})Z^{\mathrm{t}_\Lambda}_{\alpha,\beta}(q,\overline{q}) \tag{3.15}$$

agrees with the torus partition function of the diagonal $\hat{\mathfrak{g}}_k$-WZW model

$$Z^{\hat{\mathfrak{g}}_k}(q,\overline{q}) = \sum_{\hat\lambda\in\hat\Omega_k}\chi_{\hat\lambda}(q)\chi_{\hat\lambda}(\overline{q})\,. \tag{3.16}$$

The fact that the orbifold group Γ_k acts trivially on the W-algebras of coset and toroidal models makes this representation of the WZW models useful for the study of current-current deformations. Namely, for given $(\mathfrak{h},\overline{\mathfrak{h}})$ the holomorphic and antiholomorphic W-algebras $\hat{\mathfrak{h}}\subset\hat{\mathfrak{g}}_k$ and $\hat{\overline{\mathfrak{h}}}\subset\hat{\mathfrak{g}}_k$ can be identified with the $\hat{\mathrm{u}}(1)^d$, $\overline{\hat{\mathrm{u}}}(1)^d$ W-algebras of t_{Λ_k}, and deformations corresponding to $(\mathfrak{h},\overline{\mathfrak{h}})$ are then just deformations of the toroidal factor t_{Λ_k} in (3.9).

Thus, for a point $O\in\mathcal{M}_{\mathrm{WZW}}$ the corresponding conformal field theory can be represented as[4]

$$\hat{\mathfrak{g}}_k(O)\simeq\left(\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}\otimes\mathrm{t}_{\mathrm{O}\Lambda_\mathrm{k}}\right)/\Gamma_k\,. \tag{3.17}$$

This provides concrete isomorphisms between current-current deformation spaces of WZW- and toroidal models. Note however that in general the duality groups of the WZW models $\hat{\mathfrak{g}}_k$ and the toroidal models t_{Λ_k} do not agree. On the one hand, there might be dualities in the WZW models which do not preserve Λ_k and thus correspond to a change of the orbifold representation, and on the other hand dualities in the toroidal model need not lift to automorphisms of Λ_0 together with S_k. As noted above, for $\mathfrak{g}=\mathrm{su}(2)$, the duality groups of the WZW model and the corresponding toroidal model coincide.

In fact, the conformal field theories in $\mathcal{M}_{\mathrm{WZW}}$ have WZW-like sigma model descriptions, which will be discussed in section 4.

4. Sigma model considerations

4.1 Orbifold, current-current deformation and coset

To show that the deformed WZW models discussed above indeed have WZW-like sigma model descriptions, i.e. they are described by WZW-type actions, however with metrics

[4] A realization of the torus partition function of the deformed $\hat{\mathrm{su}}(2)_k$-WZW models as a partition function of $\mathbb{Z}_k$-orbifolds of tensor product of parafermionic models and free bosonic theories has already been given in [8].

different from the bi-invariant metrics and additional B-fields, we write down a sigma model action, which on the one hand defines orbifold models as in (3.17) and on the other hand describes WZW-like models.

Let us start by describing the ingredients used to construct this sigma model. As above, we denote by G a compact semi-simple Lie group of rank d with Lie algebra $\mathfrak{g}$, Cartan subgroup $\mathrm{H} \subset \mathrm{G}$, corresponding Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and $k \in \mathbb{N}$. Furthermore, Σ is a 2-dimensional surface, bounding the three manifold B. Then, an *asymmetrically gauged WZW model* on Σ is described by the following action

$$S^{\mathrm{asym}}_{\mathrm{G},k}(g,\mathcal{A},\mathcal{B},\lambda) := S^{\mathrm{WZW}}_{\mathrm{G},k}(g) + S^{\mathrm{vg}}_{\mathrm{G},k}(g,\mathcal{A}) + S^{\mathrm{ag}}_{\mathrm{G},k}(g,\mathcal{B}) + S^{\mathrm{c}}_{\mathrm{G},k}(g,\mathcal{A},\mathcal{B}) + S^{\mathrm{L}}_{\mathrm{G},k}(\lambda,\mathcal{B})\,, \tag{4.1}$$

where

$$S^{\mathrm{WZW}}_{\mathrm{G},k}(g) = -\frac{k}{4\pi}\left(\int_\Sigma d^2z\,\langle g^{-1}\partial g, g^{-1}\overline{\partial} g\rangle - \frac{1}{3}\int_B g^*\chi\right) \tag{4.2}$$

is the WZW action with $g : \Sigma \longrightarrow \mathrm{G}$, $\langle .,. \rangle$ the Killing form on $\mathfrak{g}$, and χ the three form on G associated to $\langle [.,.], . \rangle$ [2]. The integral over B in (4.2) is called *Wess-Zumino term.* (4.2) defines a quantum field theory if $H_2(G) = 0$ and $\frac{1}{12\pi}\chi \in H^3(G, 2\pi\mathbb{Z})$, which we assume to hold. This WZW model is vectorially coupled (see e.g. [20]) to an H-gauge connection $\mathcal{A} = (A, \overline{A}) \in \Omega^1(\Sigma, \mathfrak{h})$ by

$$S^{\mathrm{vg}}_{\mathrm{G},k}(g,\mathcal{A}) = \frac{k}{2\pi}\int_\Sigma d^2z\,\left\{\langle A, g^{-1}\partial_{\bar z} g\rangle - \langle \partial_z g g^{-1}, \bar A\rangle - \langle (1-\mathrm{Ad}_g)\,A, \bar A\rangle\right\}, \tag{4.3}$$

and axially coupled (see e.g. [6, 21]) to an H-gauge connection $\mathcal{B} = (B, \bar B) \in \Omega^1(\Sigma, \mathfrak{h})$ by

$$S^{\mathrm{ag}}_{\mathrm{G},k}(g,\mathcal{B}) = \frac{k}{2\pi}\int_\Sigma d^2z\,\left\{\langle B, g^{-1}\partial_{\bar z} g\rangle + \langle \partial_z g g^{-1}, \bar B\rangle - \langle (1+\mathrm{Ad}_g)\,B, \bar B\rangle\right\}. \tag{4.4}$$

Adding both terms (4.3) and (4.4) to the WZW action (4.2), one obtains the general asymmetrically gauged model, provided we introduce a coupling between the two gauge fields $\mathcal{A}$ and $\mathcal{B}$

$$S^{\mathrm{c}}_{\mathrm{G},k}(g,\mathcal{A},\mathcal{B}) = \frac{k}{2\pi}\int_\Sigma d^2z\,\left\{\langle (1+\mathrm{Ad}_g)\,B, \bar A\rangle - \langle (1+\mathrm{Ad}_g)\,A, \bar B\rangle\right\}. \tag{4.5}$$

This contains more terms than the interaction given in [22], where however a constraint relating the two gauge fields was imposed. We avoid such a constraint by introducing a Lagrange multiplier

$$S^{\mathrm{L}}_{\mathrm{G},k}(\lambda,\mathcal{B}) = \frac{ik}{\pi}\int_\Sigma d^2z\,\left\{-\langle \partial_z\lambda, \bar B\rangle + \langle B, \partial_{\bar z}\lambda\rangle\right\}, \tag{4.6}$$

with $\lambda : \Sigma \longrightarrow \mathfrak{h}/(2\pi \mathrm{Q}_l(\mathfrak{g})')$, where $\mathrm{Q}_l(\mathfrak{g})'$ is the image of $\mathrm{Q}_l(\mathfrak{g})$ under the isomorphism between $\mathfrak{h}^*$ and $\mathfrak{h}$ provided by $\langle .,. \rangle$.

This model will be coupled to an axially gauged H-WZW model, i.e. an axially gauged d-dimensional toroidal model

$$S_{\mathrm{H},k,E}(y,\mathcal{B}) = -\frac{k}{4\pi}\int_\Sigma d^2z\,\left\langle \left(y^{-1}\partial_z y - 2B\right), E\left(y^{-1}\partial_{\bar z} y - 2\bar B\right)\right\rangle, \tag{4.7}$$

where $y : \Sigma \to \mathrm{H} \simeq \mathrm{U}(1)^d$, and metric and B-field are parametrized by an invertible $d \times d$-matrix with positive definite symmetric part $E \in \mathrm{Gl}^{\mathrm{ps}}(d, \mathbb{R})$.

Below, it will be argued that the action

$$S_{\mathrm{G},k}(g, \mathcal{A}, \mathcal{B}, \lambda, y) := S^{\mathrm{asym}}_{\mathrm{G},k}(g, \mathcal{A}, \mathcal{B}, \lambda) + S_{\mathrm{H},k,E}(y, \mathcal{B}) \tag{4.8}$$

describes the current-current deformed $\hat{\mathfrak{g}}_k$-WZW models discussed in section 3. In fact the vectorially gauged G-WZW model part $S^{\mathrm{WZW}}_{\mathrm{G},k} + S^{\mathrm{vg}}_{\mathrm{G},k}$ of the action (4.8) describes the parafermionic factor in the orbifold representation (3.17) of the deformed models, while the H-part (4.7) describes the toroidal one. The axial gauging which couples these two parts in (4.8) amounts to the orbifold-construction in (3.17).

To see this, we will make use of the local symmetries of the model (4.8) under vector transformations

$$\begin{aligned}
g &\to hgh^{-1} \\
A &\to A + h\partial_z h^{-1} \\
\bar{A} &\to \bar{A} + h\partial_{\bar{z}} h^{-1} \\
\lambda &\to \lambda + \kappa \,,
\end{aligned}$$

with $h = \exp(i\kappa) : \Sigma \longrightarrow \mathrm{H}$, and axial transformations

$$\begin{aligned}
g &\to fgf \,, \\
B &\to B + f^{-1}\partial_z f \,, \\
\bar{B} &\to \bar{B} + f^{-1}\partial_{\bar{z}} f \,, \\
y &\to fyf = f^2 y = y f^2 \,,
\end{aligned} \tag{4.9}$$

with $f = \exp(i\eta), \eta : \Sigma \longrightarrow \mathfrak{h}/(\frac{1}{2}\mathrm{Q}(\mathfrak{g})')$.

Let us first integrate out the Lagrange multiplier field λ. Performing a partial integration in (4.6) yields

$$\begin{aligned}
S^{\mathrm{L}}_{\mathrm{G},k}(\lambda, \mathcal{B}) &= \frac{ik}{\pi} \int_\Sigma \left(-d\left\langle \lambda, \mathcal{B}\right\rangle + \left\langle \lambda, \mathcal{F}(\mathcal{B})\right\rangle\right) \\
&= 2ik \sum_i \left\langle n_i, \oint_{\gamma_i} \mathcal{B} \right\rangle + \frac{ik}{\pi} \int_\Sigma \left\langle \lambda, \mathcal{F}(\mathcal{B})\right\rangle \,,
\end{aligned} \tag{4.10}$$

where $\mathcal{F} = d\mathcal{B}$ is the curvature of $\mathcal{B}$, γ_i represent the non-trivial one-cycles of Σ and n_i are the "winding numbers" of λ around the dual cycles. Thus, the Lagrange multiplier forces $\mathcal{B}$ to be flat with logarithms of monodromies around γ_i[5]

$$b_i := \oint_{\gamma_i} \mathcal{B} \in (2k\mathrm{Q}_l(\mathfrak{g}))^* \simeq \frac{1}{2k}\mathrm{P}(\mathfrak{g}) \,. \tag{4.11}$$

Now, axial gauge transformations can shift the b_i by elements in $\frac{1}{2}\mathrm{Q}_l(\mathfrak{g})$. Therefore, the gauge equivalence classes of flat connections $\mathcal{B}$ satisfying (4.11) are completely characterized

[5] A similar construction was used in [23].

by $b_i \in (\frac{1}{2k}\mathrm{P}(\mathfrak{g}))/(\frac{1}{2}\mathrm{Q}_l(\mathfrak{g})) \simeq \Gamma_k$, and the integration over the gauge field $\mathcal{B}$ reduces to summing over these b_i. Note that $\Gamma_k = \mathrm{P}(\mathfrak{g})/(k\mathrm{Q}_l(\mathfrak{g}))$ is nothing else than the orbifold group in the orbifold representation of the deformed WZW models (3.17).

Having integrated out λ and $\mathcal{B}$ we end up with a product of an H-gauged G-WZW model, corresponding to the parafermionic coset model $\hat{\mathfrak{g}}_k/\hat{\mathfrak{h}}$ [20], and a d-dimensional toroidal model parametrized by E, which are coupled by Γ_k-twists. These twists indeed correspond to the Γ_k-action (3.12) on the Hilbert spaces of the conformal field theories $\hat{\mathfrak{g}}_k/\hat{h} \otimes \mathrm{t}_{O_E\Lambda_k}$, and the sigma models (4.8) describe the deformed WZW models (3.17). The identification of parametrizations of deformation spaces is as usual in toroidal CFTs: $\mathrm{O}(d,d)$ acts on $\mathrm{Gl}^{\mathrm{ps}}(d,\mathbb{R})$ by fractional linear transformations, and we get $O(\mathbf{1}) = E$ for $O \in \mathrm{O}(d,d)$ parametrizing the orbifold models (3.17) and $E \in \mathrm{Gl}^{ps}(d,\mathbb{R})$ parametrizing the sigma models (4.8).

Next, we construct the "T-dual" models by integrating out $\mathcal{B}$ first. Since $\mathcal{B}$ enters the action algebraically, this can be done by solving the equations of motion and plugging back the solution into the action. The calculation simplifies if we gauge fix $y = 1$. Solving the equations of motion for $\mathcal{B}$ yields

$$\begin{aligned} B &= \left(\mathrm{PAd}_g + 1 + 2E^T\right)^{-1} \left\{\mathrm{P}\partial_z g g^{-1} - (1+\mathrm{Ad}_g)\, A - 2\Lambda^{-1}\partial_z \Lambda\right\}, \\ \bar{B} &= \left(\mathrm{PAd}_{g^{-1}} + 1 + 2E\right)^{-1} \left\{\mathrm{P}g^{-1}\partial_{\bar{z}} g + \left(1+\mathrm{Ad}_{g^{-1}}\right) \bar{A} + 2\Lambda^{-1}\partial_{\bar{z}}\Lambda\right\}, \end{aligned} \tag{4.12}$$

where $\mathrm{P} : \mathfrak{g} \longrightarrow \mathfrak{h}$ denotes the orthogonal projection on the Cartan subalgebra and $\Lambda = \exp(i\lambda)$. This we plug back into the action (4.8). In the end, we will also integrate over $\mathcal{A}$. Therefore, we gauge fix $\Lambda = 1$ already at this stage. Then we obtain

$$\begin{aligned} S_{\mathrm{G},k}(g,\mathcal{A}) &= S^{\mathrm{WZW}}_{\mathrm{G},k}(g) + S^{\mathrm{vg}}_{\mathrm{G},k}(g,\mathcal{A}) + \frac{k}{2\pi}\int_\Sigma d^2z \left\langle \left(\mathrm{PAd}_g + 1 + 2E^T\right)^{-1}\xi, \bar{\xi}\right\rangle, \\ \xi &= \mathrm{P}\partial_z g g^{-1} - (1+\mathrm{Ad}_g)\, A, \\ \bar{\xi} &= \mathrm{P}g^{-1}\partial_{\bar{z}} g + \left(1 + \mathrm{Ad}_{g^{-1}}\right)\bar{A}. \end{aligned} \tag{4.13}$$

Next, we integrate out $\mathcal{A}$ by solving the classical equations of motion

$$\begin{aligned} A &= \left(\mathrm{PAd}_g - 1 - (1+\mathrm{PAd}_g)\left(\mathrm{PAd}_g + 1 + 2E^T\right)^{-1}(1+\mathrm{PAd}_g)\right)^{-1} \times \\ &\quad \times \left\{\mathrm{P}\partial_z g g^{-1} - (1+\mathrm{PAd}_g)\left(\mathrm{PAd}_g + 1 + 2E^T\right)^{-1}\mathrm{P}\partial_z g g^{-1}\right\}, \\ \bar{A} &= \left(\mathrm{PAd}_{g^{-1}} - 1 - \left(1+\mathrm{PAd}_{g^{-1}}\right)\left(\mathrm{PAd}_{g^{-1}} + 1 + 2E\right)^{-1}\left(1+\mathrm{PAd}_{g^{-1}}\right)\right)^{-1} \times \\ &\quad \times \left\{-\mathrm{P}g^{-1}\partial_{\bar{z}} g + \left(1+\mathrm{PAd}_{g^{-1}}\right)\left(\mathrm{PAd}_{g^{-1}} + 1 + 2E\right)^{-1}\mathrm{P}g^{-1}\partial_{\bar{z}} g\right\}. \end{aligned} \tag{4.14}$$

Plugging this back into the action and performing some algebra yields

$$\begin{aligned} S_{\mathrm{G},k}(g) &= S^{\mathrm{WZW}}_{\mathrm{G},k} + \frac{k}{2\pi}\int d^2z \left\langle \left(\mathrm{PAd}_g - R^{-1}\right)^{-1}\mathrm{P}\partial_z g g^{-1}, \mathrm{P}g^{-1}\partial_{\bar{z}} g\right\rangle \\ &= -\frac{k}{4\pi}\int_\Sigma \langle \mathcal{E}_g g^{-1}\partial_z g, g^{-1}\partial_{\bar{z}} g\rangle + \frac{k}{12\pi}\int_B g^*\chi, \end{aligned} \tag{4.15}$$

with the abbreviations

$$
\begin{aligned}
R &:= \frac{E^T - \mathrm{P}}{E^T + \mathrm{P}}\,, \\
\mathcal{E}_g &:= (1-\mathrm{P}) - \left(\mathrm{P}\mathrm{Ad}_g - R^{-1}\right)^{-1}\left(\mathrm{P}\mathrm{Ad}_g + R^{-1}\right) \\
&= (1-\mathrm{P}) - \left(R\mathrm{P}\mathrm{Ad}_g - \mathrm{P}\right)^{-1}\left(R\mathrm{P}\mathrm{Ad}_g + \mathrm{P}\right).
\end{aligned} \tag{4.16}
$$

Thus, we obtain the action of a WZW-like model with a deformed metric and additional B-field encoded in the choice of $\mathfrak{h} \subset \mathfrak{g}$ and of the matrix E. As expected from the comparison with the CFT considerations we recover the action of the original G-WZW model for $E = \mathbf{1}$.

Note that in general deformed metric and B-field are not bi-invariant with respect to G. But they are bi-invariant with respect to $\mathrm{H} \subset \mathrm{G}$, which follows from the identities $\mathcal{E}_{hg} = \mathcal{E}_g$, $\mathcal{E}_{gh} = \mathrm{Ad}_h^{-1}\mathcal{E}_g\mathrm{Ad}_h$ for all $g \in \mathrm{G}$, $h \in \mathrm{H}$.

Moreover, as also expected from the CFT results, for generic E the model (4.8) only has a $\mathfrak{h} \oplus \mathfrak{h}$ chiral symmetry algebra which is generated by

$$
\delta g = \epsilon g - gR\epsilon\,, \qquad \bar{\delta} g = g\bar{\epsilon} - R^T\bar{\epsilon}g\,, \tag{4.17}
$$

with $\epsilon, \bar{\epsilon} : \Sigma \longrightarrow \mathfrak{h}$. The corresponding chiral currents read

$$
\begin{aligned}
J &= kR^{-1}\left(1 - RR^T\right)\left(\mathrm{P}\mathrm{Ad}_g - R^{-1}\right)^{-1}\mathrm{P}\partial_z g g^{-1}\,, \\
\bar{J} &= -k\left(R^T\right)^{-1}\left(1 - R^T R\right)\left(\mathrm{P}\mathrm{Ad}_{g^{-1}} - \left(R^T\right)^{-1}\right)^{-1}\mathrm{P}g^{-1}\partial_{\bar{z}} g\,.
\end{aligned} \tag{4.18}
$$

Let us mention that the two degenerations of the model (4.8) $E = \lambda\mathbf{1}$, $\lambda \to 0$, $\lambda \to \infty$ correspond to the axially and the vectorially gauged WZW model respectively.[6] Thus, at the classical level, we achieved to construct a class of models connecting the axially gauged WZW model via the ungauged to the vectorially gauged one.

Moreover, the response of our sigma model (4.15) to a variation of the deformation parameters

$$
\begin{aligned}
\delta S &= \frac{k}{2\pi}\int d^2z \left\langle \left(\delta R^{-1}\right)\left(\mathrm{P}\mathrm{Ad}_g - R^{-1}\right)^{-1}\mathrm{P}\partial_z g g^{-1}, \left(\mathrm{P}\mathrm{Ad}_{g^{-1}} - \left(R^T\right)^{-1}\right)^{-1}\mathrm{P}g^{-1}\partial_{\bar{z}} g \right\rangle \\
&= \frac{1}{2\pi k}\int d^2z \left\langle R\left(R^T R - 1\right)^{-1}\left(\delta R^{-1}\right)\left(1 - RR^T\right)^{-1} RJ, \bar{J} \right\rangle
\end{aligned} \tag{4.19}
$$

is bilinear in the conserved currents (4.18), suggesting that our family is indeed generated by current-current perturbations.

So far, we have seen that the action of the deformed model looks like a WZW model action with deformed bilinear form, which is generically not bi-invariant anymore. This

[6] This can be easily seen by comparing the sigma model actions. Alternatively, it can be deduced by the following observation. For $E = 0$ the additional $\mathrm{U}(1)^d$ factor decouples and integrating over $\mathcal{B}$ yields the "T-dualized" orbifold of the vectorially gauged model, i.e. the axially gauged model. On the other hand in the limit $\lambda \to \infty$ the gauge field $\mathcal{B}$ is frozen to zero and we are left with the vectorially gauged model. In both cases there is an additional decoupled $\mathrm{U}(1)^d$ factor whose torus is of vanishing size or decompactified, respectively. In our previous discussion this additional factor appears, because in the decoupling limits the gauge fixing conditions need to be altered, i.e. in those limits one should gauge fix coordinates on G such that the metric on the coset does not degenerate.

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however is not the full story. When integrating out the gauge fields, one picks up jacobians which usually give rise to a non-trivial dilaton. Hence, we expect that in addition to the deformed metric and B field, there will be a non-trivial dilaton. By construction our model should be conformally invariant in the semiclassical limit. This means that the background should satisfy beta function conditions (see e.g. the review [24] and references therein). Checking these equations one will also observe that a non trivial dilaton (coupling with the power of k^0 to the sigma model action) is needed. That would be one way to obtain the non-trivial dilaton. In the following subsection, we will use a different method to derive the expression for the dilaton.

But before coming to that, let us comment on the relation of these deformed models to the families

$$S_{\tilde{E}}^{(\mathrm{G}\times\mathrm{H})/\mathrm{H}}(g,\mathcal{B},y) := S_{\mathrm{G},k}^{WZW}(g) + S_{\mathrm{G},k}^{\mathrm{ag}}(g,\mathcal{B}) + S_{\mathrm{H},k,\tilde{E}}(y,\mathcal{B}) \tag{4.20}$$

of gauged WZW models of type $(\mathrm{G}\times\mathrm{H})/\mathrm{H}$ with varying embedding of the gauge group in the symmetry group of the G- and H-WZW models, parametrized by $\tilde{E}\in \mathrm{Gl}^{ps}(d,\mathbb{R})$. For the special case of symmetric $\tilde{E}$ these models were described in [25].[7]

Integrating out the gauge fields $\mathcal{B}$ in these models, one obtains[8]

$$S_{\tilde{E}}^{(\mathrm{G}\times\mathrm{H})/\mathrm{H}}(g) = S_{\mathrm{G},k}^{WZW}(g) + \frac{k}{2\pi}\int d^2z \left\langle \left(\mathrm{PAd}_g + 1 + 2\tilde{E}^T\right)^{-1} \mathrm{P}\partial_z g g^{-1}, \mathrm{P}g^{-1}\partial_{\bar{z}} g \right\rangle . \tag{4.21}$$

This in fact coincides with the action (4.15), which was obtained by integrating out $\mathcal{B}$ in the sigma model (4.8) we started with, if one relates the parameters

$$\tilde{E} = -E\,(E-1)^{-1} \ . \tag{4.22}$$

Hence, for E such that $\tilde{E}$ in (4.22) is well-defined and positive definite, the sigma model (4.8) we started with has a coset realization given by (4.20). This is the case e.g. for $E\in \mathrm{Gl}^{ps}(d,\mathbb{R})$ whose eigenvalues lie in $(0,1)$. Note however that e.g. for $E=1$, $\tilde{E}$ in (4.22) is not well-defined and thus, the original G-WZW model does not have a proper coset realization as in (4.20). It only corresponds to a degeneration thereof. Nevertheless, for convenience, we will use the coset model realization in the following subsection to calculate the hamiltonian of the model (4.8). Although the realization we use is only defined on part of the actual moduli space, we suppose that the result we obtain is actually valid on the whole moduli space. This is supported by the observation that it reproduces the hamiltonian of the WZW model at $E=\mathbf{1}$.

4.2 Hamiltonian and dilaton shift

Next, we would like to derive the hamiltonian of the coset models (4.20). In order to perform the Legendre transform, we return to minkowskian worldsheet signature, i.e. $z\to$

[7] In fact, the concept of constructing families of sigma-models as gauged models with varying embedding of the gauge group is very general in nature and has been applied e.g. to the case of WZW-models corresponding to non-compact Lie groups in [26, 27].

[8] The result can be read off from (4.12) by setting $A=\bar{A}=0$, $E\mapsto\tilde{E}$ and $\Lambda=1$.

$x^+ = \tau + \sigma$, $\bar{z} \to x^- = \tau - \sigma$, $\partial_{z/\bar{z}} \to \partial_{+/-} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ and $d^2z \to 2d^2\sigma$. Let us first discuss the ungauged G-WZW model (4.2). The conjugate momenta are given by

$$\varpi^T = \left(\frac{\delta S}{\delta g^{-1}\partial_\tau g}\right)^T = -\frac{k}{4\pi} g^{-1}\partial_\tau g + \Omega\,, \tag{4.23}$$

where Ω denotes the contribution from the Wess-Zumino term. Since it contains exactly one τ derivative, its final effect drops out in the hamiltonian. The hamiltonian is obtained by performing the Legendre transform

$$\begin{aligned} H &= \int d\sigma\, \mathrm{Tr}\left(\varpi^T g^{-1}\partial_\tau g\right) - \int d\sigma L \\ &= -\frac{k}{8\pi}\int d\sigma \left(\langle g^{-1}\partial_\tau g, g^{-1}\partial_\tau g\rangle + \langle g^{-1}\partial_\sigma g, g^{-1}\partial_\sigma g\rangle\right), \end{aligned} \tag{4.24}$$

where L stands for the lagrangian belonging to (4.2). Introducing the currents

$$J_+ = -k\partial_+ g g^{-1}\,, \qquad J_- = k g^{-1}\partial_- g \tag{4.25}$$

we arrive at the classical Sugawara expression

$$H = -\frac{1}{4\pi k}\int d\sigma \left(\langle J_+, J_+\rangle + \langle J_-, J_-\rangle\right). \tag{4.26}$$

For later use, we also give these currents expressed as phase space functions

$$\begin{aligned} J_+ &= 2\pi \mathrm{Ad}_g \varpi^T - 2\pi k \mathrm{Ad}_g \Omega - \frac{k}{2}\partial_\sigma g g^{-1}\,, \\ J_- &= -2\pi k \varpi^T + 2\pi k \Omega - \frac{k}{2} g^{-1}\partial_\sigma g\,. \end{aligned} \tag{4.27}$$

In order to construct the hamiltonian of the deformed G-WZW models we follow the prescription given in [28], where our starting point will be the coset model (4.20) before integrating out the gauge fields, but after gauge fixing $y = 1$, i.e.

$$\begin{aligned} S_{\tilde{E}}^{(\mathrm{G}\times\mathrm{H})/\mathrm{H}}(g, \mathcal{B}) = S^{WZW}(g) + \frac{k}{\pi}\int d^2\sigma \Big\{ &\langle B_+, g^{-1}\partial_- g\rangle + \langle \partial_+ g g^{-1}, B_-\rangle - \\ &- \Big\langle \Big(1 + \mathrm{Ad}_g + 2\tilde{E}^T\Big) B_+, B_- \Big\rangle \Big\}. \end{aligned} \tag{4.28}$$

The additional terms as compared to the ungauged model modify the conjugate momenta according to

$$\varpi^T \longmapsto \varpi^T + \frac{k}{2\pi}B_+ + \frac{k}{2\pi}\mathrm{Ad}_g B_-\,. \tag{4.29}$$

The corresponding hamiltonian reads

$$\begin{aligned} \mathcal{H} = H - \frac{k}{2\pi}\int d\sigma \Big(&- \langle B_+, g^{-1}\partial_\sigma g\rangle + \langle \partial_\sigma g g^{-1}, B_-\rangle - \\ &- 2\Big\langle \Big(1 + \mathrm{Ad}_g + 2\tilde{E}^T\Big) B_+, B_- \Big\rangle \Big), \end{aligned} \tag{4.30}$$

where H denotes the hamiltonian (4.26) of the ungauged model.

Next, we modify the currents $J_\pm$, such that the modified currents

$$\begin{aligned}\mathcal{J}_+ &= J_+ + k\mathrm{PAd}_g B_+ + kB_- \,,\\ \mathcal{J}_- &= J_- - kB_+ - k\mathrm{PAd}_{g^{-1}} B_- \,,\end{aligned} \tag{4.31}$$

obey a gauge invariant Poisson algebra (see [28] for more details). In terms of the new currents the hamiltonian of the deformed model is given by

$$\begin{aligned}\mathcal{H} = -\frac{1}{4k\pi}\int d\sigma\Big(&\langle\mathcal{J}_+,\mathcal{J}_+\rangle + \langle\mathcal{J}_-,\mathcal{J}_-\rangle - 4k\langle B_-,\mathcal{J}_+\rangle + 4k\langle B_+,\mathcal{J}_-\rangle +\\ &+ 2k^2\langle B_+,B_+\rangle + 2k^2\langle B_-,B_-\rangle - 4k^2\left\langle B_+,\left(1+2\tilde{E}\right)B_-\right\rangle\Big)\,.\end{aligned} \tag{4.32}$$

The additional constraints due to the vanishing of the conjugate momenta of $B_\pm$ can be used to eliminate the gauge fields by solving their algebraic equations of motion

$$\left(1+2\tilde{E}^T\right)B_+ - B_- = -\frac{1}{k}\mathrm{P}\mathcal{J}_+, \tag{4.33}$$

$$\left(1+2\tilde{E}\right)B_- - B_+ = \frac{1}{k}\mathrm{P}\mathcal{J}_- \,. \tag{4.34}$$

For $\tilde{E}$ invertible[9] these equations allow a unique solution for $B_\pm$. Before giving this it is useful to employ (4.33) and (4.34) to simplify the expression (4.32) slightly. For the last term we write

$$2\left\langle\left(1+2\tilde{E}^T\right)B_+,B_-\right\rangle = \left\langle\left(-\frac{1}{k}\mathrm{P}\mathcal{J}_+ + B_-\right),B_-\right\rangle + \left\langle B_+,\left(\frac{1}{k}\mathrm{P}\mathcal{J}_- + B_+\right)\right\rangle$$

and obtain

$$\mathcal{H} = -\frac{1}{4\pi k}\int d\sigma\Big(\langle\mathcal{J}_+,\mathcal{J}_+\rangle + \langle\mathcal{J}_-,\mathcal{J}_-\rangle + 2k\langle B_+,\mathcal{J}_-\rangle - 2k\langle B_-,\mathcal{J}_+\rangle\Big)\,. \tag{4.35}$$

Finally, we plug in the solutions of the equations of motion (4.33), (4.34) of $\mathcal{B}$

$$\begin{aligned}B_+ &= \frac{1}{2k}\left(\tilde{E}+\tilde{E}^T+2\tilde{E}\tilde{E}^T\right)^{-1}\left(\mathrm{P}\mathcal{J}_- - \left(1+2\tilde{E}\right)\mathrm{P}\mathcal{J}_+\right),\\ B_- &= \frac{1}{2k}\left(\tilde{E}+\tilde{E}^T+2\tilde{E}^T\tilde{E}\right)^{-1}\left(\left(1+2\tilde{E}^T\right)\mathrm{P}\mathcal{J}_- - 2\mathrm{P}\mathcal{J}_+\right).\end{aligned} \tag{4.36}$$

The result for the hamiltonian of the deformed model is

$$\begin{aligned}\mathcal{H} = -\frac{1}{4\pi k}\int d\sigma\Bigg(&\left\langle\left(1+\mathrm{P}\left(\tilde{E}+\tilde{E}^T+2\tilde{E}^T\tilde{E}\right)^{-1}\mathrm{P}\right)\mathcal{J}_+,\mathcal{J}_+\right\rangle +\\ &+\left\langle\left(1+\mathrm{P}\left(\tilde{E}+\tilde{E}^T+2\tilde{E}\tilde{E}^T\right)^{-1}\mathrm{P}\right)\mathcal{J}_-,\mathcal{J}_-\right\rangle -\\ &-\Big\langle\Big(\left(\tilde{E}+\tilde{E}^T+2\tilde{E}\tilde{E}^T\right)^{-1}\left(1+2\tilde{E}\right) +\\ &+\left(1+2\tilde{E}\right)\left(\tilde{E}+\tilde{E}^T+2\tilde{E}^T\tilde{E}\right)^{-1}\Big)\mathrm{P}\mathcal{J}_+,\mathrm{P}\mathcal{J}_-\Big\rangle\Bigg)\,.\end{aligned} \tag{4.37}$$

For symmetric $\tilde{E}$ this expression agrees with the one given in [25].

[9]The degeneration $\tilde{E}=0$ for example describes the coset model G/H which is discussed in [28].

Now, the zero mode part of the hamiltonian restricted to a certain subspace of the Hilbert space should "match" with the generalized laplacian on the space of smooth functions on G [13]

$$\Delta^{\Phi} := -\frac{e^{2\Phi}}{\sqrt{\det(G)}}\partial_\mu e^{-2\Phi}\sqrt{\det(G)}G^{\mu\nu}\partial_\nu\,, \tag{4.38}$$

which takes into account the dilaton Φ. This can be explained (for more details see [13]) by the correspondence of the constraint

$$\left(L_0+\bar{L}_0-a\right)|\text{physical}\rangle = 0 \tag{4.39}$$

($L_0+\bar{L}_0$ is the hamiltonian and a is a normal ordering constant) with the mass shell condition, which in location space reads

$$\frac{e^{2\Phi}}{\sqrt{\det(G)}}\partial_\mu\left(e^{-2\Phi}\sqrt{\det(G)}G^{\mu\nu}\partial_\nu\Psi\right) = -m^2\Psi\,. \tag{4.40}$$

Here Ψ denotes the target space field associated to the physical state in (4.39). This will be used in the following to determine the dilaton in the deformed G-WZW models.

The subspace of the Hilbert space which should correspond to the space of functions on G consists of highest weight vectors only (see [29] for a discussion of this point), and thus the hamiltonian (4.37) restricted to this subspace can be written completely in terms of zero-modes of left and right currents. Since zero-modes of left and right chiral currents are the generators of left- and inverse right- multiplication by G on itself, they can be identified with the respective sections j_L and j_R of $\mathrm{TG}\otimes\mathfrak{g}^*$. Choosing a basis of $\mathfrak{g}^*$, one obtains from these sections the vector fields j_L^A, j_R^A, $A\in\{1,\ldots,\dim(\mathrm{G})\}$ on G. In every point $p\in\mathrm{G}$, $((j_L^A)_p)_A$, $((j_R^A)_p)_A$ are two basis of the tangent space $\mathrm{T}_p\mathrm{G}$ of G in p. Hence, Δ^Φ can be written in terms of j_L^A or j_R^A, Φ and the target space metric G. G can be read off from the kinetic term of the action (4.15), such that we can compare Δ^Φ with the Hamilton operator (4.37) to obtain Φ. This is the general strategy presented in [13].

But before applying it to the models (4.8), let us illustrate it in the example of the "undeformed" G-WZW model. For notational convenience, we will use the sections j_L and j_R of $\mathrm{TG}\otimes\mathfrak{g}^*$ in the following computations and denote the dual Killing form on $\mathfrak{g}^*$ by $\langle .,. \rangle$. Now, the target space metric is $2\pi k$ times the bi-invariant metric induced by the Killing form. The Laplace operator can then be written as

$$\Delta = \frac{1}{k}\langle j_L, j_L\rangle = \frac{1}{k}\langle j_R, j_R\rangle = \frac{1}{2k}\left(\langle j_L, j_L\rangle + \langle j_R, j_R\rangle\right), \tag{4.41}$$

where the second equality reflects the bi-invariance of the metric.

Observing that under the identification $J_0^A\sim j_L^A$, $\bar{J}_0^A\sim j_R^A$, of zero-modes of the holomorphic and antiholomorphic currents with generators of the left and right multiplication the hamiltonian (4.26) "matches" the laplacian Δ, we deduce that the dilaton Φ is constant.

Now let us come to the discussion of the deformed models (4.8). The hamiltonian corresponding to the coset representation of these models has been calculated above (4.37).

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The target space metric can be obtained from (4.21) by symmetrization[10]

$$
\begin{aligned}
G_{\mu\nu}/2\pi k &= \langle M j_{L\mu}, j_{L\nu}\rangle \\
&= \langle N j_{R\mu}, j_{R\nu}\rangle\,,
\end{aligned} \tag{4.42}
$$

with

$$
\begin{aligned}
M &= \left(1{+}2\tilde{E}^T\mathrm{P}{+}\mathrm{P}\mathrm{Ad}_g\mathrm{P}\right)^{-1}\left(1{-}\mathrm{P}{+}2\tilde{E}^T\mathrm{P}{+}2\tilde{E}\mathrm{P}{+}4\tilde{E}^T\tilde{E}\mathrm{P}\right)\times \\
&\quad\times\left(1{+}2\tilde{E}\mathrm{P}{+}\mathrm{P}\mathrm{Ad}_{g^{-1}}\mathrm{P}\right)^{-1}, \\
N &= \left(1{+}2\tilde{E}\mathrm{P}{+}\mathrm{P}\mathrm{Ad}_{g^{-1}}\mathrm{P}\right)^{-1}\left(1{-}\mathrm{P}{+}2\tilde{E}^T\mathrm{P}{+}2\tilde{E}\mathrm{P}{+}4\tilde{E}\tilde{E}^T\mathrm{P}\right)\times \\
&\quad\times\left(1{+}2\tilde{E}^T\mathrm{P}{+}\mathrm{P}\mathrm{Ad}_g\mathrm{P}\right)^{-1}.
\end{aligned} \tag{4.43}
$$

Now, the generalized laplacian can be written as

$$
4\pi k\Delta^{\Phi} = \left\langle f^{-1}j_L, fM^{-1}j_L\right\rangle + \left\langle f^{-1}j_R, fN^{-1}j_R\right\rangle\,, \tag{4.44}
$$

where derivatives act on everything appearing to their right, and

$$
f = e^{-2\Phi}\sqrt{\det(G)}/\sqrt{\det(G_0)}\,, \tag{4.45}
$$

with G_0 denoting the "undeformed", i.e. the Killing metric on G. The expression for the inverse of M is

$$
\begin{aligned}
2M^{-1} &= 2+\mathrm{P}\left(\tilde{E}^T+\tilde{E}+2\tilde{E}\tilde{E}^T\right)^{-1}\mathrm{P}+\mathrm{P}\left(1+2\tilde{E}\right)\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)^{-1}\mathrm{P}\mathrm{Ad}_g\mathrm{P}+ \\
&\quad+\mathrm{P}\mathrm{Ad}_{g^{-1}}\mathrm{P}\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)^{-1}\left(1+2\tilde{E}^T\right)\mathrm{P}+ \\
&\quad+\mathrm{P}\mathrm{Ad}_{g^{-1}}\mathrm{P}\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)^{-1}\mathrm{P}\mathrm{Ad}_g\mathrm{P}.
\end{aligned} \tag{4.46}
$$

In order to obtain this result we have performed a couple of manipulations. First, we employed that

$$
\left(1{+}2\tilde{E}^T\mathrm{P}{+}\mathrm{P}\mathrm{Ad}_g\mathrm{P}\right)^{-1}(1{-}\mathrm{P}) = (1{-}\mathrm{P})\left(1{+}2\tilde{E}\mathrm{P}{+}\mathrm{P}\mathrm{Ad}_{g^{-1}}\mathrm{P}\right)^{-1} = 1-\mathrm{P}\,,
$$

which follows from P being a projector. Then M takes a block diagonal form, and we have inverted the two blocks ($1-P$ and the rest) on the corresponding subspaces. Furthermore,

[10] As noted before the actions (4.8) and (4.21) agree under the identification (4.22), and we will use the parametrization by $\tilde{E}$ for the calculation of Δ^{Φ}, because the hamiltonian (4.37) was obtained in the coset representation. This however does not restrict the region of validity of the results.

on the Cartan subalgebra $\mathfrak{h}$ (where P=1) we used the following identities

$$\begin{aligned}
&-1+\frac{1}{2}\left(1+2\tilde{E}\right)\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)^{-1}\left(1+2\tilde{E}^T\right)= \\
&\quad=\frac{1}{2}\left(1+2\tilde{E}\right)\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)^{-1}\times \\
&\quad\quad\times\left(-2\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)+\left(1+2\tilde{E}^T\right)\left(1+2\tilde{E}\right)\right)\left(1+2\tilde{E}\right)^{-1} \\
&\quad=\frac{1}{2}\left(1+2\tilde{E}\right)\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)^{-1}\left(1+2\tilde{E}\right)^{-1} \\
&\quad=\frac{1}{2}\left(\tilde{E}^T+\tilde{E}+2\tilde{E}\tilde{E}^T\right)^{-1}.
\end{aligned} \tag{4.47}$$

The last line of (4.47) is most easily checked with the inverted expression

$$\begin{aligned}
\left(1+2\tilde{E}\right)\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)\left(1+2\tilde{E}\right)^{-1} &= \left(1+2\tilde{E}\right)\tilde{E}^T+\tilde{E} \\
&= \tilde{E}^T+\tilde{E}+2\tilde{E}\tilde{E}^T.
\end{aligned} \tag{4.48}$$

Analogously, one finds

$$\begin{aligned}
2N^{-1} &= 2+\mathrm{P}\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)^{-1}\mathrm{P}+\mathrm{P}\left(1+2\tilde{E}^T\right)\left(\tilde{E}^T+\tilde{E}+2\tilde{E}\tilde{E}^T\right)^{-1}\times \\
&\quad\times\mathrm{PAd}_{g^{-1}}\mathrm{P}+\mathrm{PAd}_g\mathrm{P}\left(\tilde{E}^T+\tilde{E}+2\tilde{E}\tilde{E}^T\right)^{-1}\left(1+2\tilde{E}\right)\mathrm{P}+ \\
&\quad+\mathrm{PAd}_g\mathrm{P}\left(\tilde{E}^T+\tilde{E}+2\tilde{E}\tilde{E}^T\right)^{-1}\mathrm{PAd}_{g^{-1}}\mathrm{P}.
\end{aligned} \tag{4.49}$$

In order to write down the result for the laplacian, we also need the relations

$$\mathrm{Ad}_g j_L=-j_R\,,\qquad \mathrm{Ad}_{g^{-1}}j_R=-j_L\,.$$

Collecting everything, we finally obtain

$$\begin{aligned}
4\pi k\Delta^\Phi &= \left\langle f^{-1}j_L, f\left(1+\mathrm{P}\left(\tilde{E}^T+\tilde{E}+2\tilde{E}\tilde{E}^T\right)^{-1}\mathrm{P}\right)j_L\right\rangle+ \\
&\quad+\left\langle f^{-1}j_R, f\left(1+\mathrm{P}\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)^{-1}\mathrm{P}\right)j_R\right\rangle- \\
&\quad-\frac{1}{2}\Big\langle f^{-1}\mathrm{P}j_L, f\Big(\left(1+2\tilde{E}\right)\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)^{-1}+ \\
&\qquad\qquad+\left(\tilde{E}^T+\tilde{E}+2\tilde{E}\tilde{E}^T\right)^{-1}\left(1+2\tilde{E}\right)\Big)\mathrm{P}j_R\Big\rangle- \\
&\quad-\frac{1}{2}\Big\langle f^{-1}\mathrm{P}j_R, f\Big(\left(1+2\tilde{E}^T\right)\left(\tilde{E}^T+\tilde{E}+2\tilde{E}\tilde{E}^T\right)^{-1}+ \\
&\qquad\qquad+\left(\tilde{E}^T+\tilde{E}+2\tilde{E}^T\tilde{E}\right)^{-1}\left(1+2\tilde{E}^T\right)\Big)\mathrm{P}j_L\Big\rangle.
\end{aligned} \tag{4.50}$$

This expression matches with the hamiltonian (4.37) provided that f is equal to a constant, which can be taken to one by performing a constant dilaton shift. Thus, we conclude that along the deformations there is a non trivial dilaton such that

$$\sqrt{\det(G)}e^{-2\Phi} \qquad \text{is independent of } \tilde{E}. \tag{4.51}$$

The independence of this 'string measure' on the deformation parameter(s) has been observed before for one dimensional deformations [3]–[5] and for symmetric $\tilde{E}$ [25].

As a byproduct we see from (4.50) that eigenfunctions of Δ^Φ at a given $\tilde{E}$, which are also eigenfunctions with respect to the H-actions induced by the left and right multiplication of H on G are also eigenfunctions of Δ^Φ at all $\tilde{E}$, with different eigenvalues however. This is expected from the CFT considerations above, where the only effect of the deformations were changes of the $\mathfrak{h}$, $\overline{\mathfrak{h}}$ charges, and $\hat{\mathfrak{h}}$-, $\hat{\overline{\mathfrak{h}}}$-highest weight states remained highest weight states under the deformations.

To complete the discussion we should recall that the coset description we used here to construct the hamiltonian (4.37) is only valid in the part of the parameter space of the model (4.8), where $\tilde{E}$ is positive definite. In particular, it breaks down, when one of the eigenvalues of E becomes 1. Nevertheless, the hamiltonian, we obtained, can be continued to the region where E has eigenvalues 1, as can be read off from

$$\begin{aligned}\mathcal{H} = -\frac{1}{4\pi k}\int d\sigma\Big(&\Big\langle\Big(1+\mathrm{P}\left(E-1\right)\left(E+E^T\right)^{-1}\left(E^T-1\right)\mathrm{P}\Big)\mathcal{J}_+,\mathcal{J}_+\Big\rangle + \\ &+\Big\langle\Big(1+\mathrm{P}\left(E^T-1\right)\left(E+E^T\right)^{-1}\left(E-1\right)\mathrm{P}\Big)\mathcal{J}_-,\mathcal{J}_-\Big\rangle + \\ &+2\Big\langle\left(E^T-1\right)\left(E+E^T\right)^{-1}\left(E+1\right)\mathrm{P}\mathcal{J}_+,\mathrm{P}\mathcal{J}_-\Big\rangle\Big)\,,\end{aligned} \tag{4.52}$$

and in fact for $E=\mathbf{1}$ it coincides with the one of the G-WZW model (4.26). This suggests that (4.52) is indeed the hamiltonian on the whole parameter space of (4.8) and all the results from this section also apply to the entire moduli space.

4.3 The SU(2) example

In this subsection we would like to illustrate our previous discussion very explicitly in the simplest example, namely G = SU(2). We do not want to go through all the details, since much of the discussion for SU(2) (or $\mathrm{SL}(2,\mathbb{R})$)[11] can be found in the literature (e.g. in [3]–[5]), but we will address two topics here, which were important in the previous sections. Firstly, we will derive the deformed sigma model by taking the T-dual of the model $(\mathrm{SU}(2)_k/\,\mathrm{U}(1)\times\mathrm{U}(1))\,/\mathbb{Z}_k$, and secondly we will compute the spectrum of the generalized laplacian.

From the discussion of current-current deformed WZW models corresponding to arbitrary compact semi-simple Lie groups in section 3, we know that the deformed $\hat{\mathrm{su}}(2)_k$-WZW models can be realized as orbifold models (compare (3.17))

$$\left(\hat{\mathrm{su}}(2)_k/\hat{\mathrm{u}}(1)\otimes\hat{\mathrm{u}}(1)_{\sqrt{k}R}\right)/\mathbb{Z}_k\,, \tag{4.53}$$

[11] The discussion of deformation is often presented for the non compact version of A_1 because in that case the interesting phenomenon of smooth topology change is observed.

where $R \in (0,\infty)$ parametrizes the $\hat{\mathrm{u}}(1)$-factor.[12] That the deformed $\hat{\mathrm{su}}(2)_k$-WZW model can be written in this way has first been suggested by Yang in [8], where a one-parameter family of modular invariant partition functions corresponding to such orbifold models with varying radius in the $\hat{\mathrm{u}}(1)$-factor was presented, noting that the partition function at $R = 1$ coincides with the partition function of the $\hat{\mathrm{su}}(2)_k$-WZW model [30].

In the following we will compare this with explicit sigma model analysis. We start with the $(\mathrm{SU}(2)/\,\mathrm{U}(1)\times\mathrm{U}(1))/\mathbb{Z}_k$ gauged sigma model (the parametrization is taken from [4])[13]

$$S = \frac{k}{2\pi}\int d^2z\left\{\partial_+ x\partial_- x + \tan^2 x\partial_+\theta\partial_-\theta + \frac{1}{R^2}\partial_+ y\partial_- y\right\}, \tag{4.54}$$

where for the time being we omitted the dilaton term coupling to the Gauss-Bonnet density. Now, we redefine coordinates according to

$$\theta = \alpha + \beta\,, \qquad y = \alpha - \beta\,. \tag{4.55}$$

In these coordinates the relevant components of the target space metric can be written as

$$\begin{aligned} G_{\alpha\alpha} &= G_{\beta\beta} = k\left(\tan^2 x + \frac{1}{R^2}\right) \\ G_{\alpha\beta} &= k\left(\tan^2 x - \frac{1}{R^2}\right). \end{aligned} \tag{4.56}$$

Now we T-dualize the α direction. The T-dual metric and B-field follow from the Buscher fromulæ [31]. We obtain

$$\begin{aligned} \tilde{G}_{\tilde{\alpha}\tilde{\alpha}} &= \frac{1}{G_{\alpha\alpha}} = \frac{R^2\cos^2 x}{k\left(\cos^2 x + R^2\sin^2 x\right)} \\ \tilde{G}_{\beta\beta} &= G_{\beta\beta} - \frac{G_{\alpha\beta}^2}{G_{\alpha\alpha}} = \frac{k\sin^2 x}{\cos^2 x + R^2\sin^2 x} \\ \tilde{G}_{\tilde{\alpha}\beta} &= \frac{B_{\alpha\beta}}{G_{\alpha\alpha}} = 0 \\ \tilde{B}_{\tilde{\alpha}\beta} &= \frac{G_{\alpha\beta}}{G_{\alpha\alpha}} = -2\frac{\cos^2 x}{\cos^2 x + R^2\sin^2 x} + 1\,. \end{aligned} \tag{4.57}$$

Now we define $\tilde{\alpha} = k\tilde{\theta}$, $\beta = \theta$ and drop the constant term 1 in the B-field. This amounts to

$$\begin{aligned} S^R = \frac{k}{2\pi}\int d^2z\Bigg\{&\partial_+ x\partial_- x + \frac{\sin^2 x}{\cos^2 x + R^2\sin^2 x}\partial_+\theta\partial_-\theta + \frac{R^2\cos^2 x}{\cos^2 x + R^2\sin^2 x}\partial_+\tilde{\theta}\partial_-\tilde{\theta} + \\ &+ \frac{\cos^2 x}{\cos^2 x + R^2\sin^2 x}\left(\partial_+\theta\partial_-\tilde{\theta} - \partial_+\tilde{\theta}\partial_-\theta\right)\Bigg\}\,. \end{aligned} \tag{4.58}$$

[12] As alluded to above, $R \mapsto \frac{1}{R}$ is a duality.

[13] In order to make contact with our general discussion in subsection 4.1 we note that in [4] an SU(2) group element is written as $\exp[i(\theta-\tilde{\theta})\sigma_3/2]\exp[ix\sigma_2]\exp[i(\theta+\tilde{\theta})\sigma_3/2]$. The dependence on $\tilde{\theta}$ is removed after performing the vector gauging. Constant shifts in θ correspond to the axial symmetry. The orbifold action of $\mathbb{Z}_k$ will be specified below in the discussion after (4.59). It essentially reduces the radius of a circle lying diagonally in the θy-torus.

It remains to discuss the periodicity of $\tilde{\theta}$. In order to obtain the deformed model $\tilde{\theta}$ should be a 2π periodic coordinate, i.e. $\tilde{\alpha}$ should be a $2\pi k$ periodic coordinate. If we perform the T-duality according to the prescription given in [23], the intermediate gauged sigma model on a worldsheet Σ will contain a term (compare (4.10))

$$S_{wind} = \sum_i n_i \oint_{\gamma_i} A \,, \tag{4.59}$$

where γ_i are one cycles of Σ and n_i are the winding numbers of the Lagrange multiplier $\tilde{\alpha}$ around one cycles of Σ dual to γ_i. As in the discussion around (4.10), summing over the $k\mathbb{Z}$ valued windings of $\tilde{\alpha}$ yields a Kronecker delta, which is non-vanishing if $\oint A$ takes values in $2\pi\mathbb{Z}/k$. Since the original model is obtained by absorbing a pure gauge $A_\pm = \partial_\pm \rho$ into a redefinition of the original coordinate, α parameterises actually an orbifold $S^1/\mathbb{Z}_k$ where S^1 is the unit circle.

Thus for the case $\mathrm{G} = \mathrm{SU}(2)$ we explicitly obtained the deformed WZW action (4.15) from the orbifold representation $\left(\mathrm{SU}(2)_k/\,\mathrm{U}(1)\times\mathrm{U}(1)\right)/\mathbb{Z}_k$.

Next, we would like to discuss the generalized laplacian Δ^Φ in this example. For notational simplicity we set $k=1$ during the calculation and reinstall it in the end. After a coordinate change

$$\rho = \sin x \tag{4.60}$$

the metric of the deformed model takes the form

$$ds^2 = \frac{d\rho^2}{1-\rho^2} + \frac{\rho^2}{1+(R^2-1)\,\rho^2} d\theta^2 + \frac{R^2\left(1-\rho^2\right)}{1+(R^2-1)\,\rho^2} d\tilde{\theta}^2 \,. \tag{4.61}$$

Up to a constant shift, the dilaton is given by the relation

$$e^{-2\Phi}\sqrt{\det(G)} = \rho \,. \tag{4.62}$$

Thus, we find explicitly

$$\Delta^\Phi_R = \frac{e^{2\Phi}}{\sqrt{\det(G)}}\partial_\mu e^{-2\Phi}\sqrt{\det(G)}G^{\mu\nu}\partial_\nu = \Delta_{R=1} + (R^2-1)\partial_\theta^2 + \frac{(1-R^2)}{R^2}\partial_{\tilde{\theta}}^2 \,, \tag{4.63}$$

which shows that eigenfunctions of the Laplace operator corresponding to the Killing metric on SU(2) which are also eigenfunctions of ∂_θ, $\partial_{\tilde{\theta}}$ are in fact eigenfunctions of the generalized laplacian Δ^Φ for all R,[14] however with different eigenvalues. For an eigenfunction of the Laplace operator in the irreducible SU(2) representation labelled by $j \in \{0,\ldots,\frac{k}{2}\}$ with left and right U(1)-quantum numbers n and $\bar{n}$, the difference of the eigenvalues of Δ^Φ at R and $R=1$ is given by

$$\delta^k_{j,n,m}(R) = \frac{R^2-1}{4k}n^2 + \frac{1-R^2}{4kR^2}\bar{n}^2 \,, \tag{4.64}$$

where we have reintroduced the level k.

[14] Note that at $R=1$ $\Delta^\Phi = \Delta$ as discussed in the previous section.

Now, let us compare this with the CFT description (3.17). The difference of $\frac{1}{2}(L_0+\bar{L}_0)$-eigenvalues of a highest weight state in the r-twisted sector for R and $R=1$ can be calculated to be

$$d^k_{j,r,p} = \frac{1}{4k}\left(\frac{1-R^2}{R^2}m^2 + \left(R^2-1\right)(m-r)^2\right), \tag{4.65}$$

where we already identified the moduli spaces according to our general discussion.

We observe that (4.64) and (4.65) agree, if we identify

$$n = m\,, \qquad \bar{n} = m-r\,, \tag{4.66}$$

and this identification is actually the one we expected from the $\mathbb{Z}_k$-action (3.12).

5. Discussion

Having shown that the effect of current-current deformations of a conformal field theory on its structure is completely captured by deformations of a charge lattice, we obtained a description of the subspaces of CFT moduli spaces corresponding to these deformations as moduli spaces of certain lattices with additional structure. This generalizes the case of deformations of toroidal conformal field theories [10].

The general considerations were applied to WZW models, where they were compared with a realization of the deformed models as orbifolds of products of coset models with varying toroidal models. This realization was well suited for the construction of sigma models corresponding to the deformed WZW models. For this purpose we employed axial-vector duality to transform the orbifold of the $(G/H\times H)$-sigma model into a WZW-like model with (in general) non-bi-invariant metric, additional B-field and constant dilaton. This provides a very explicit description of the sigma models associated to deformed WZW models. It would be interesting to investigate further the geometry of metric and B-field we obtained for the deformed models. Since the sigma models correspond to conformal field theories, they should for example satisfy some nice differential equations, namely the beta-function equations (see [24] and references therein).

Apart from the general CFT considerations, we focused the discussion on the example of WZW models associated to compact, semi-simple Lie groups. There are however other interesting conformal field theories admitting current-current deformations, which deserve an analysis of their moduli spaces. These are for example WZW models corresponding to non-semi-simple or non-compact Lie-groups (see e.g. [32]–[36]). The former possess more complicated moduli spaces than semi-simple WZW models, because they give rise to more than one inequivalent deformation spaces, while the latter could provide time dependent exact string backgrounds and thus might give hints about how string theory deals with cosmological singularities (see e.g. [37]–[39] for a discussion of this point). It should be easy to modify the presented discussion in such a way that one can obtain the models of Guadagnini et al. (see e.g. [40]–[43] for discussions of those models) as limits in a class of deformed theories. One interesting aspect could be that these models can be viewed as coset models with a trivial dilaton. Beyond a generalization to arbitrary WZW-models, also an investigation of moduli spaces of e.g. coset models, and in particular Kazama-Suzuki-models [44] should be of interest, because the latter would provide examples of explicitly known moduli spaces of $N = 2$ superconformal field theories. A discussion of the relation between mirror symmetry and gauge symmetry in this setting has been presented in [45].

The analysis of the "behaviour" of D-branes (i.e. conformal boundary conditions) on moduli spaces of conformal field theories was in fact the main motivation for this study of current-current deformations. As our considerations show, these deformations are in fact easily tractable, and hence provide a good setup to study "bulk-deformations" of boundary conformal field theories. Some semi-classical aspects of D-branes in deformed SU(2)-WZW models were presented in [46, 47]. The general conformal field theory analysis of boundary conditions in deformed WZW models will be addressed in a forthcoming publication [48].

Acknowledgments

The authors thank Matthias Gaberdiel, Werner Nahm, Jacek Pawelczyk, Andreas Recknagel and Katrin Wendland for useful discussions. S.F. is grateful for the kind hospitality extended towards him during a visit at Warsaw University where he had been given the opportunity to present some of the results reported in this article. Especially, he would like to thank Zygmunt Lalak. D.R. was supported by DFG Schwerpunktprogramm 1096 and by the Marie Curie Training Site "Strings, Branes and Boundary Conformal Field Theory" at King's College London, under EU grant HPMT-CT-2001-00296. The work of S.F. is supported in part by the European Community's Human Potential Programme under contracts HPRN-CT-2000-00131 Quantum Spacetime, HPRN-CT-2000-00148 Physics Across the Present Energy Frontier and HPRN-CT-2000-00152 Supersymmetry and the Early Universe, and INTAS 00-561.

A. Deformation theory

In this appendix, techniques from conformal deformation theory (see e.g. [10] and [14]–[16]) are used to calculate the effect of current-current deformations on arbitrary conformal field theories containing current algebras in their holomorphic and antiholomorphic W-algebras.

In the following, a family of conformal field theories is regarded as a hermitean vector bundle over a differentiable manifold[15] parametrizing deformations of the conformal field theory structures in a smooth way, i.e. all CFT structures are smooth sections of corresponding vector bundles.

Such families can be realized as perturbations (2.1) by exactly marginal fields. In this case, the tangent bundle of their base manifolds are subbundles of the hermitean vector bundles. The choice of regularization method and renormalization scheme gives rise to connections on them [16]. Here connections D (called $\overline{c}$ in [16]) will be used, which restrict to the Levi-Civita connections on the tangent bundles of the base manifolds equipped with the respective Zamolodchikov metrics. These connections are defined by "minimal subtraction" of divergences in the regularization constant.

[15] Singularities do not occur in our situation.

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Given a conformal field theory, it is in general quite hard to make global statements about the family of conformal field theories, generated by perturbation with a given set of exactly marginal fields. This is due to the fact that information on the CFT structures in points of the family have to be more or less completely reconstructed out of the structures at one point (the point corresponding to the CFT which is being perturbed), by means of perturbation theory. Thus, one gets perturbative results only, and perturbation theory usually becomes technically difficult at higher orders.

However, in the case of perturbations by products of holomorphic and antiholomorphic currents, there is in fact enough structure to make exact global statements about the families of CFTs generated by them using first order perturbation theory only.

In [1] it was shown that tensor products of fields of holomorphic and antiholomorphic currents are exactly marginal, if and only if they form abelian current algebras $\hat{\mathfrak{a}}$, $\hat{\overline{\mathfrak{a}}}$ respectively, and that in this case the deformations generated by them preserve the corresponding current algebras $\hat{\mathfrak{a}}$ and $\hat{\overline{\mathfrak{a}}}$. Thus these deformations give rise to families of conformal field theories with $\hat{\mathfrak{a}}$ and $\hat{\overline{\mathfrak{a}}}$ contained in their holomorphic and antiholomorphic W-algebras. Moreover, the tangent vectors to the families in every point are given by products of currents, whose CFT-properties are known independently of the actual CFT. Thus the derivatives of the CFT-structures can be calculated in every point of the families and can then be integrated up.

Assuming that the Hilbert spaces of the conformal field theories in the families decompose into $\hat{\mathfrak{a}} \times \hat{\overline{\mathfrak{a}}}$-highest weight representations as in (2.4)

$$\mathcal{H} \simeq \bigoplus_{(Q,\overline{Q})\in\Lambda} \mathcal{H}_{Q\overline{Q}} \otimes \mathcal{V}_Q \otimes \overline{\mathcal{V}}_{\overline{Q}} ,$$

it is shown in the following that these deformations only affect the $\hat{\mathfrak{a}} \times \hat{\overline{\mathfrak{a}}}$- representations, while the OPE-coefficients of $\hat{\mathfrak{a}} \times \hat{\overline{\mathfrak{a}}}$-highest weight states are parallel with respect to the connection D. To be more precise, the only effect of the deformations will be $\mathrm{O}(d,\overline{d})$-transformations of the charge lattices $\Lambda \subset \mathfrak{a}^* \times \overline{\mathfrak{a}}^*$. From this it follows in particular that the corresponding deformation spaces are given by (2.7)

$$\mathcal{D}_{(a,\overline{a})} \simeq \mathrm{O}(d,\overline{d})_0/(\mathrm{O}(d)\times\mathrm{O}(\overline{d}))_0 \simeq \mathrm{O}(d,\overline{d})/\mathrm{O}(d)\times\mathrm{O}(\overline{d}) . \tag{A.1}$$

Let us first of all calculate the covariant derivatives of the modes of the $\hat{\mathfrak{a}}$- and $\hat{\overline{\mathfrak{a}}}$-currents and holomorphic and antiholomorphic energy-momentum tensors defined by

$$\begin{aligned} j^\alpha(z) &= \sum_{n\in\mathbb{Z}} z^{n-1} j_n^\alpha , & \overline{\jmath}^{\overline{\alpha}}(\overline{z}) &= \sum_{n\in\mathbb{Z}} \overline{z}^{n-1} \overline{\jmath}_n^{\overline{\alpha}} , \\ T(z) &= \sum_{n\in\mathbb{Z}} z^{n-2} L_n , & \overline{T}(\overline{z}) &= \sum_{n\in\mathbb{Z}} \overline{z}^{n-2} \overline{L}_n , \end{aligned} \tag{A.2}$$

where, as in section 2 $(j^\alpha)_\alpha$ and $(\overline{\jmath}^{\overline{\alpha}})_{\overline{\alpha}}$ are basis of $\mathfrak{a}$ and $\overline{\mathfrak{a}}$ respectively, and we denote the generators of the deformations by $\mathcal{O}^{\alpha\overline{\alpha}}(z,\overline{z}) := j^\alpha(z)\overline{\jmath}^{\overline{\alpha}}(\overline{z})$. By the definition of D, $D_{\mathcal{O}^{\alpha\overline{\alpha}}} j_n^\beta$ can be expressed as

$$D_{\mathcal{O}^{\alpha\overline{\alpha}}} j_n^\beta = \left[\oint_{C(0)} \frac{\mathrm{d}z}{2\pi i} z^{-n} \int_{\mathbb{C}P^1\setminus D_\epsilon(z)} \mathrm{d}^2 w O_{\alpha\overline{\alpha}}(w,\overline{w}) j^\beta(z) \right]_\epsilon , \tag{A.3}$$

where ϵ is the regularization parameter and $[X]_\epsilon$ means the regularization independent part of X. Using the OPE (2.2) this can be written as

$$\begin{aligned}
D_{\mathcal{O}^{\alpha\overline{\alpha}}} j_n^\beta &= \left[\oint_{C(0)} \frac{\mathrm{d}z}{2\pi i} z^{-n} \oint_{C_\epsilon(z)} \frac{\mathrm{d}\overline{w}}{-2i} \frac{kK^{\alpha\beta}\overline{\jmath}^{\overline{\alpha}}(\overline{w})}{(z-w)}\right]_\epsilon \\
&= \left[\oint_{C_\epsilon(0)} \frac{\mathrm{d}\overline{w}}{2i} kK^{\alpha\beta}\overline{\jmath}^{\overline{\alpha}}(\overline{w}) \oint_{C(w)} \frac{\mathrm{d}z}{2\pi i} \frac{z^{-n}}{(z-w)}\right]_\epsilon \\
&= \left[\oint_{C_\epsilon(0)} \frac{\mathrm{d}\overline{w}}{2i} kK^{\alpha\beta}\overline{\jmath}^{\overline{\alpha}}(\overline{w}) w^{-n}\right]_\epsilon \\
&= -\pi k K^{\alpha\beta}\overline{\jmath}_0^{\overline{\alpha}}\delta_{n,0}\,. \qquad\qquad (A.4)
\end{aligned}$$

The same kind of arguments lead to

$$D_{\mathcal{O}^{\alpha\overline{\alpha}}} L_n^\beta = \left[\oint_{C(0)} \frac{\mathrm{d}z}{2\pi i} z^{-n} \int_{\mathbb{C}P^1 \setminus D_\epsilon(z)} \mathrm{d}^2 w O_{\alpha\overline{\alpha}}(w,\overline{w}) T(z)\right]_\epsilon = -\pi j_n^\alpha \overline{\jmath}_0^{\overline{\alpha}} \qquad (A.5)$$

and similar expressions for the modes of $\hat{\overline{\mathfrak{a}}}$. Altogether we find

$$\begin{aligned}
D_{\mathcal{O}^{\alpha\overline{\alpha}}} j_n^\beta &= -\pi k K^{\alpha\beta}\delta_{n,0}\overline{\jmath}_n^{\overline{\alpha}}\,, \qquad & D_{\mathcal{O}^{\alpha\overline{\alpha}}} L_n &= -\pi j_n^\alpha \overline{\jmath}_0^{\overline{\alpha}}\,, \\
D_{\mathcal{O}^{\alpha\overline{\alpha}}} \overline{\jmath}_n^{\overline{\beta}} &= -\pi k \overline{K}^{\overline{\alpha}\overline{\beta}}\delta_{n,0} j_n^\alpha\,, \qquad & D_{\mathcal{O}^{\alpha\overline{\alpha}}} \overline{L}_n &= -\pi j_0^\alpha \overline{\jmath}_n^{\overline{\alpha}}\,. \qquad (A.6)
\end{aligned}$$

Thus, the $\hat{\mathfrak{a}} \times \hat{\overline{\mathfrak{a}}}$-$W$-algebra structure is parallel with respect to D, and only the $\mathfrak{a} \times \overline{\mathfrak{a}}$-charges[16] change under the deformations. Their covariant derivatives can be read off from (A.6)

$$D_{c_{\alpha\overline{\alpha}}\mathcal{O}^{\alpha\overline{\alpha}}}\left(Q^\beta, \overline{Q}^{\overline{\beta}}\right) = -\pi\left(c_\alpha^{\overline{\beta}} Q^\alpha, c_{\overline{\alpha}}^{\beta}\overline{Q}^{\overline{\alpha}}\right), \qquad (A.7)$$

which are transformations in $o(\mathfrak{a}^* \oplus \overline{\mathfrak{a}}^*, \kappa - \overline{\kappa})$. This characterizes completely the deformations of the $\hat{\mathfrak{a}} \times \hat{\overline{\mathfrak{a}}}$-$W$-algebra structures of the CFTs. Moreover the $\hat{\mathfrak{a}} \times \hat{\overline{\mathfrak{a}}}$-highest weight property and the decomposition of the Hilbert space (2.4) are preserved under the deformations, which means in particular that we only have to assume (2.4) for one CFT in the family.

To show that this is in fact the only effect of the deformations on the CFT structures, we have to show, that the OPE of $\hat{\mathfrak{a}} \times \hat{\overline{\mathfrak{a}}}$-highest weight vectors is not deformed. Now, the

[16]As noted above, we assume the zero modes of the currents to be diagonalizable on $\mathcal{H}$.

covariant derivative of correlation functions of those vectors is given by

$$\begin{aligned}
&D_{\mathcal{O}^{\alpha\overline{\alpha}}} \langle \Phi_1(z_1,\overline{z}_1)\ldots\Phi_n(z_n,\overline{z}_n)\rangle = \\
&\quad= \left[\int_{\mathbb{C}P^1\setminus\bigcup_i D_\epsilon(z_i)} \mathrm{d}^2 w \,\langle \mathcal{O}^{\alpha\overline{\alpha}}\Phi_1(z_1,\overline{z}_1)\ldots\Phi_n(z_n,\overline{z}_n)\rangle\right]_\epsilon \\
&\quad= \sum_{i,j} Q_i^\alpha \overline{Q}_j^{\overline{\alpha}} \langle \Phi_1(z_1,\overline{z}_1)\ldots\Phi_n(z_n,\overline{z}_n)\rangle \left[\int_{\mathbb{C}P^1\setminus\bigcup_i D_\epsilon(z_i)} \frac{-1}{(w-z_i)(\overline{w}-\overline{z}_j)}\right]_\epsilon \\
&\quad= -2\pi \sum_{i<j} Q_i^\alpha \overline{Q}_j^{\overline{\alpha}} ln\left(|z_i-z_j|^2\right)\langle \Phi_1(z_1,\overline{z}_1)\ldots\Phi_n(z_n,\overline{z}_n)\rangle \qquad \text{(A.8)} \\
&\quad= \left(D_{\mathcal{O}^{\alpha\overline{\alpha}}} ln\left(\prod_{i<j}(z_i-z_j)^{2\kappa(Q_i,Q_j)}(\overline{z}_i-\overline{z}_j)^{2\overline{\kappa}(\overline{Q}_i,\overline{Q}_j)}\right)\right)\langle \Phi_1(z_1,\overline{z}_1)\ldots\Phi_n(z_n,\overline{z}_n)\rangle\,.
\end{aligned}$$

But the ln-term in the last line of (A.8) is just the logarithm of the corresponding $\hat{\mathfrak{a}}$-, $\hat{\overline{\mathfrak{a}}}$-conformal block. Thus, the correlation functions are deformed only through the conformal blocks and the OPE-coefficients of $\hat{\mathfrak{a}}\times\hat{\overline{\mathfrak{a}}}$-highest weight states are parallel.

Thus, the effect of current-current deformations on the conformal field theory structure is completely characterized by the deformations of the charges described above. In particular from (A.7) it follows that the base manifold of the family of CFTs generated by current-current deformations is indeed given by (A.1).

Let us finish with a comment on another connection $\tilde{D}$. In the discussion above, we used connections D on the Hilbert space bundles over the deformation spaces, which were defined by minimal subtraction. With respect to these connections, operators from the current algebra orthogonal to the zero modes are parallel. For the discussion of e.g. boundary conditions other connections $\tilde{D}$ will be useful. These are defined by[17]

$$\tilde{D}_{\mathcal{O}_{\alpha\overline{\alpha}}} := D_{\mathcal{O}_{\alpha\overline{\alpha}}} - \pi\sum_{n\neq 0}\frac{1}{n} j_n^\alpha \overline{\jmath}_n^{\overline{\alpha}}\,. \qquad \text{(A.9)}$$

They satisfy

$$\begin{aligned}
\tilde{D}_{\mathcal{O}^{\alpha\overline{\alpha}}} j_n^\beta &= -\pi k K^{\alpha\beta}\overline{\jmath}_{-n}^{\overline{\alpha}}\,, &\qquad \tilde{D}_{\mathcal{O}^{\alpha\overline{\alpha}}} L_n &= -\pi\textstyle\sum_m j_{m+n}^\alpha \overline{\jmath}_m^{\overline{\alpha}}\,,\\
\tilde{D}_{\mathcal{O}^{\alpha\overline{\alpha}}} \overline{\jmath}_n^{\overline{\beta}} &= -\pi k \overline{K}^{\overline{\alpha}\overline{\beta}} j_{-n}^\alpha\,, &\qquad \tilde{D}_{\mathcal{O}^{\alpha\overline{\alpha}}} \overline{L}_n &= -\pi\textstyle\sum_m j_m^\alpha \overline{\jmath}_{m+n}^{\overline{\alpha}}\,, \qquad \text{(A.10)}
\end{aligned}$$

and thus for all $n\in\mathbb{Z}$, the parallel transport of $(j_n^\alpha,\overline{\jmath}_{-n}^{\overline{\alpha}})_{\alpha,\overline{\alpha}}$ is given by the vector representation of $\mathrm{O}(d,\overline{d})$. As the connections D, also $\tilde{D}$ restrict to the Levi-Civita-connection on the tangent bundle of the deformation space equipped with the Zamolodchikov metric.

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